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Principles of Analog Computation

PRINCIPLES OF ANALOG COMPUTATION

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PRINCIPLES OF ANALOG COMPUTATION

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Preface

The authors' aim has been to present, in a single volume, those methods of analog computing which form the fundamental tools for the analyst working in the field. They have endeavored to do this by emphasizing analog-computation techniques. In general the plan has been to present only enough discussion of component principles to afford an understanding of their operation. In doing this the component devices are introduced, and the mathematical operation performed by a particular device is stressed. The main emphasis of the book has been placed upon the type of problems which can be solved, and it is hoped that the techniques presented will prove to be helpful to a wide variety of users. It is the authors' belief that the main utility of the analog computer has been, and will be, in the fields of servomechanisms and particle dynamics. Thus they make no apologies for drawing heavily from these fields for examples. In this regard it is hoped that the research worker in these fields will find the analog computer an indispensable tool. For, having mastered its use, the researcher need not restrict his efforts to approximations which he can solve mathematically but is free to investigate the true facts, whatever they may be. This allows the analyst to focus his attention on the problem to be solved without becoming lost in a jungle of numerical gymnastics.

For those interested in the detailed design of analog-computer components, this book will not prove to be adequate. To such persons we apologize, but we hope that by reading this book they may find ideas which will aid them in appreciating the operator's problems, and thus that the design of future equipment will make more generous allowance for these problems than in the past.

As a textbook the material of this book has been used as the basis of a course in analog computations presented for UCLA extension in the San Diego area. It has been found that a background including basic electronics and Laplace-transformation theory is required to master this course successfully. With this background it has been found that senior or first-year graduate students in the physical sciences experienced little

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difficulty in mastering the material of the text. In this regard it is felt that working the problems at the end of the chapters forms an integral part of the course.

In writing any book authors must draw from many sources for the material they present. It is impossible for us to acknowledge by name most of those to whom we are indebted; however, our thanks are given to those persons in the field through association with whom the information in this text has been developed. In particular we should like to express our appreciation to W. F. Uplinger and W. Abern of the NAMTC Guidance and Control Laboratory, Point Mugu, Calif. We should also like to thank Mrs. Ann Phillips and Miss Terry Jenkins for their help in preparing the manuscript. The cooperation of the Ryan Aeronautical Company during the project is likewise appreciated.

George W. Smith
Roger C. Wood

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CHAPTER 1

Major Components of the Analog Computer

1-1. INTRODUCTION

The word analog is derived from the Greek *analogos*, meaning "according to a due ratio or proportion." Hence the analog-computer family consists of all those devices in which measurable physical quantities are made to obey mathematical relationships comparable with those existing in a particular problem. Slide rules, a-c network analyzers, electronic and mechanical differential analyzers, modulated carrier electronic computers, and other such devices are all analog computers. For the purposes of this text, however, we shall restrict the name analog computer to those devices which utilize the d-c operational amplifier as the basic component. In particular we shall discuss the electronic differential analyzer.

In the strictest sense, an analog computer does not compute. It is simply an electronic model of a given physical system. This model is constructed in the laboratory, where, under carefully controlled conditions, measurements are taken of parameters of interest. The computer consists of a collection of basic building blocks which can be interconnected so that they are governed by the same set of equations as those describing the system to be analyzed. In the electronic differential analyzer the components are capable of summation, integration, multiplication, and arbitrary-function generation. It thus is capable of all the operations necessary to solve ordinary linear and nonlinear differential equations. In the computers which we shall consider, the variables are represented by d-c voltages, with time generally used as the independent variable.

1-2. OPERATIONAL AMPLIFIERS

The fundamental component of the electronic differential analyzer is the d-c operational amplifier, which is the basis of adders, integrators, etc. The input and feedback impedances of the amplifier determine the mathematical operation it performs on the input signal. The operational amplifier consists of a high-gain d-c amplifier with input impedance Z_I and feedback impedance Z_F.

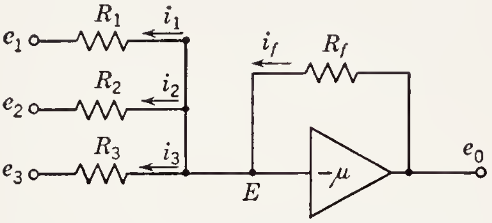

FIG. 1-1. Operational amplifier with resistors as input and feedback impedances.

If the input and feedback elements of an amplifier are resistors, the output will be the sum of the inputs, with an inversion in sign and possible multiplication factors, as determined by the ratio of the feedback resistor to the input resistor. Consider the circuit of Fig. 1-1. In this circuit the triangular symbol is used to designate the d-c operational amplifier. Let μ be the amplifier forward gain, which will be large over the range of operating frequencies. The voltage at the amplifier input grid is denoted by E, so that

$$e_o = -\mu E \qquad \text{or} \qquad E = -\frac{e_o}{\mu} \tag{1-1}$$

In such high-gain amplifiers the input grid is practically at ground potential, while the output voltage has a finite value. According to Kirchhoff's first law, the sum of all currents flowing to the grid must be zero. Thus

$$i_1 + i_2 + i_3 + i_f = 0^* \tag{1-2}$$

or, from Ohm's law,

$$\frac{e_1 - E}{R_1} + \frac{e_2 - E}{R_2} + \frac{e_3 - E}{R_3} - \frac{E - e_o}{R_f} = 0 \tag{1-3}$$

Substituting for E and solving for e_o, we obtain

$$e_o = \frac{-R_f}{1 + \frac{1}{\mu}\left(1 + \frac{R_f}{R_1} + \frac{R_f}{R_2} + \frac{R_f}{R_3}\right)}\left(\frac{e_1}{R_1} + \frac{e_2}{R_2} + \frac{e_3}{R_3}\right) \tag{1-4}$$

In most practical amplifiers the amplifier gain μ is of the order of 10^8 and the ratio $R_f/R_i < 1{,}000$. Therefore, in Eq. (1-4), we are justified in taking the limit as $\mu \to \infty$, to obtain

$$e_o = -R_f\left(\frac{e_1}{R_1} + \frac{e_2}{R_2} + \frac{e_3}{R_3}\right) \tag{1-5}$$

---

* Grid current into the d-c amplifier proper is negligible (this amounts to neglecting the grid current in the first stage of vacuum-tube amplification). In most practical designs this current is of the order of magnitude of 10^{-11} amp.

In the general case for n input voltages the performance equation is

$$e_o = -R_f\left(\sum_{i=1}^{n} \frac{e_i}{R_i}\right) \tag{1-6}$$

which is the general equation for the operation of the summing amplifier. It is particularly useful for establishing relations of the form

$$e_o = a_1e_1 + a_2e_2 + \cdots + a_ne_n \tag{1-7}$$

with $$a_1 = \frac{R_f}{R_1} \qquad a_2 = \frac{R_f}{R_2} \qquad \cdots \qquad a_n = \frac{R_f}{R_n}$$

The quantities $a_1 = R_f/R_1$, $a_2 = R_f/R_2$, . . . , are referred to as the gain associated with the voltages e_1, e_2, . . . , respectively. A commonly used summing amplifier (REAC)* employs the following arrangement:

$$\begin{aligned} R_f &= 1 \text{ megohm} \\ R_1 = R_2 = R_3 = R_4 &= 1 \text{ megohm} \\ R_5 = R_6 &= 0.25 \text{ megohm} \\ R_7 = R_8 &= 0.10 \text{ megohm} \end{aligned}$$

thus providing four gains of 1, two gains of 4, and two gains of 10. Some machines make a distinction between a summing amplifier and an inverting amplifier. The term inverting amplifier is used for those amplifiers whose inputs are restricted to gains of 1.

At this point, it is worth noting a difference in philosophy which exists as to wiring of analog computers. In the computers provided by some manufacturers each amplifier has the input and summing resistors permanently wired to it (REAC, PACE, EASE). This provides the operator with a collection of fixed gains and eliminates the necessity of external wiring. Gains other than those provided are obtained by means of potentiometers. The other school of thought provides only a limited number of permanently wired components (GEDA, EASE) but furnishes external resistors in a wide range of values. This necessitates the operator externally wiring in the gains which he desires. Of the two schools of thought, the former offers a great deal of simplicity at the expense of a certain amount of flexibility, while the latter features flexibility at the expense of simplicity. It is the authors' opinion that the optimum arrangement would be a combination of the two. That is, provide each amplifier with a set of standard gains, and, in addition, make provisions for each amplifier to have a summing junction associated with it for putting in external resistors. Also, each amplifier should be wired so that the feedback resistor can be disconnected at the discretion of the operator. The most recent trend has been to place all the passive network computing elements in a temperature-controlled oven. This is done to provide increased accuracy but does not necessarily do away with plug-in components.

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* In this text the trademarked abbreviations REAC, PACE, GEDA, EASE are used to designate the equipment manufactured by Reeves Instrument Company, Electronics Associates, Inc., Goodyear Aircraft Company, Beckman Instrument Company, respectively.

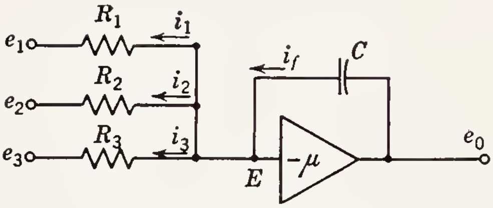

FIG. 1-2. Operational amplifier with capacitor feedback and resistance inputs.

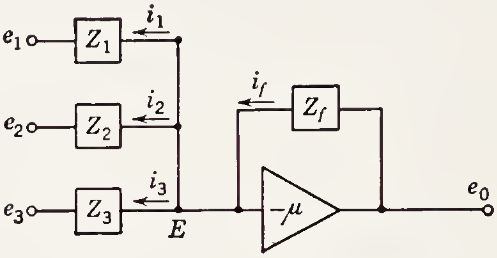

FIG. 1-3. Operational amplifier with general impedance feedback and general impedance inputs.

If the feedback resistor is replaced by a capacitor (Fig. 1-2), the amplifier becomes an integrator. In this case the nodal equation becomes

$$\frac{E_1 + E_o/\mu}{R_1} + \frac{E_2 + E_o/\mu}{R_2} + \frac{E_3 + E_o/\mu}{R_3} = \left(E_o + \frac{E_o}{\mu}\right) Cs^* \qquad (1\text{-}8)$$

Solving for E_o, we obtain

$$E_o = \frac{-1/Cs}{1 + \frac{1}{\mu}\left(1 + \frac{1}{R_1Cs} + \frac{1}{R_2Cs} + \frac{1}{R_3Cs}\right)} \left(\frac{E_1}{R_1} + \frac{E_2}{R_2} + \frac{E_3}{R_3}\right) \qquad (1\text{-}9)$$

The above equation reduces in the limit when $\mu \to \infty$ to

$$E_o = -\frac{1}{Cs}\left(\frac{E_1}{R_1} + \frac{E_2}{R_2} + \frac{E_3}{R_3}\right) \qquad (1\text{-}10)$$

In the general case for n input voltages,

$$E_o = -\frac{1}{Cs}\left(\sum_{i=1}^{n} \frac{E_i}{R_i}\right) \qquad (1\text{-}11)$$

which is the general equation for the operation of the integrating amplifier.

Generalizing the figure shown for the summer and integrator by making all the elements a general impedance, we obtain the circuit of Fig. 1-3. Considering the figure as an amplifier having the voltages e_1, e_2, and e_3 applied to the input grid through the impedances Z_1, Z_2, and Z_3, respectively, and the impedance Z_f connected as feedback around the amplifier, as before the nodal equation is

---

* Throughout the text capital letters are used to indicate functions of the complex frequency s, while the small letter indicates functions of time.

$$\frac{E_1 + E_o/\mu}{Z_1} + \frac{E_2 + E_o/\mu}{Z_2} + \frac{E_3 + E_o/\mu}{Z_3} = -\frac{E_o + E_o/\mu}{Z_f} \tag{1-12}$$

Solving Eq. (1-12) for E_o, we obtain

$$E_o = \frac{-Z_f}{1 + \frac{1}{\mu}\left(1 + \frac{Z_f}{Z_1} + \frac{Z_f}{Z_2} + \frac{Z_f}{Z_3}\right)}\left(\frac{E_1}{Z_1} + \frac{E_2}{Z_2} + \frac{E_3}{Z_3}\right) \tag{1-13}$$

Thus, as $\mu \longrightarrow \infty$, we obtain

$$E_o = -Z_f\left(\frac{E_1}{Z_1} + \frac{E_2}{Z_2} + \frac{E_3}{Z_3}\right) \tag{1-14}$$

In the general case for n input voltages

$$E_o = -Z_f\left(\sum_{i=1}^{n} \frac{E_i}{Z_i}\right) \tag{1-15}$$

An operational amplifier with its feedback loop opened is called a high-gain amplifier. The output of a high-gain amplifier is minus μ (the amplifier gain) times the sum of the inputs. High-gain amplifiers are employed most frequently in implicit-function techniques. For example, assume that we have a function $Z = f(X_1,X_2, \ldots ,X_n)$, in which Z is either difficult or impossible to compute, while an equivalent implicit function $R(X_1,X_2, \ldots ,X_n,Z) = 0$ offers no such complication. Examples of such functions are $Z = X_1/X_2$ and $R(X_1,X_2,Z) = ZX_2 - X_1 = 0$;

$$Z = \sqrt{X_1^2 + X_2^2} \qquad \text{and} \qquad R(X_1,X_2,Z) = Z^2 - X_1^2 - X_2^2 = 0$$

$Z = \tan^{-1} X_1/X_2$ and $R(X_1,X_2,Z) = X_2 \sin Z - X_1 \cos Z = 0$. If R is fed into a high-gain amplifier whose output Z is used in the computation of R, it can be seen that the amplifier now has a feedback loop closed through the circuit generating R. Chapter 6 gives the theory for using high-gain amplifier circuits.

When a d-c amplifier is used as an integrator, relay circuits must be employed to prevent integration of the input signals until the operate control is activated. This permits the introduction of initial-condition voltages across the feedback capacitors and allows the operator to stop (or hold) integration once it has started. For the present, it is sufficient to say that these items can be accomplished. A detailed discussion of operating relays and initial conditions is contained in Chap. 8.

The normal computing amplifier is designed to put out a maximum current when operating at the extremes of the designed computing voltage ranges. If the load requirements are such that a greater current than the designed maximum is called for, the output tube of the amplifier is cut off. To get the additional current required, the output impedance can be changed by reducing the plate-load resistance in the output stage. This can be conveniently done, without modifying the amplifier itself, by making simple connections in the patch bay and on the problem board. If a small resistor (10 to 20 kilohms) is connected from the amplifier output to the output-tube plate-voltage power supply, the amplifier can be made to deliver two to three times its normal load current. This is known as boosting the amplifier. If a boost is applied to the output of an amplifier, there is a reduction in the gain of the amplifier. In most high-quality chopper stabilized amplifiers, however, this reduction in gain is not serious.

1-3. POTENTIOMETERS

Potentiometers are adjustable voltage dividers so arranged that if a voltage e_i is applied to one end, and the other end grounded, a voltage e_o will be read at the arm which is in a constant ratio to the input voltage e_i. This ratio $e_o/e_i = a$ has a value which varies from 0 to 1. Figure 1-4 shows the circuit diagram for a potentiometer and the equivalent symbol commonly used to represent a potentiometer in programming the analog computer. The coefficient setting may be read on the potentiometer dial. The dial scale will be linear unless the potentiometer is loaded appreciably by another computing element connected to its output terminals. If the potentiometer is loaded, then corrections must be applied to the dial reading to obtain the correct setting.

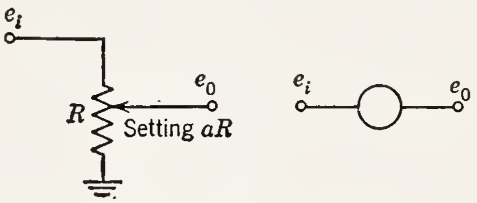

FIG. 1-4. Coefficient potentiometer and equivalent symbol.

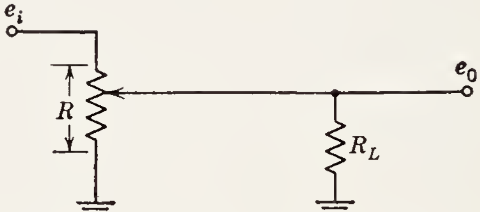

FIG. 1-5. Potentiometer with resistance load R_L.

Ordinarily, the dial reading a of a computing potentiometer refers to the fraction aR of the total potentiometer resistance R across which the output voltage e is measured. If a load resistance R_L is connected to the potentiometer output terminals (see Fig. 1-5), the above equation no longer holds. From the nodal equation for the output terminal, we have by Kirchhoff's first law

$$(e_o - e_i)\,\frac{1}{(1-a)R} + \frac{e_o}{aR} + \frac{e_o}{R_L} = 0 \tag{1-16}$$

Solving for the ratio e_o/e_i, we obtain

$$\frac{e_o}{e_i} = \frac{a}{1 + a(1 - a)R/R_L} \tag{1-17}$$

This is the correct relation between the potentiometer setting a and the transfer function e_o/e_i. The quantity

$$a - \frac{e_o}{e_1} = a - \frac{a}{1 + a(1 - a)R/R_L} \tag{1-18}$$

is a measure for the error made by applying the simple transfer function instead of the correct relationship. A plot of potentiometer loading error is shown in Fig. 1-6.

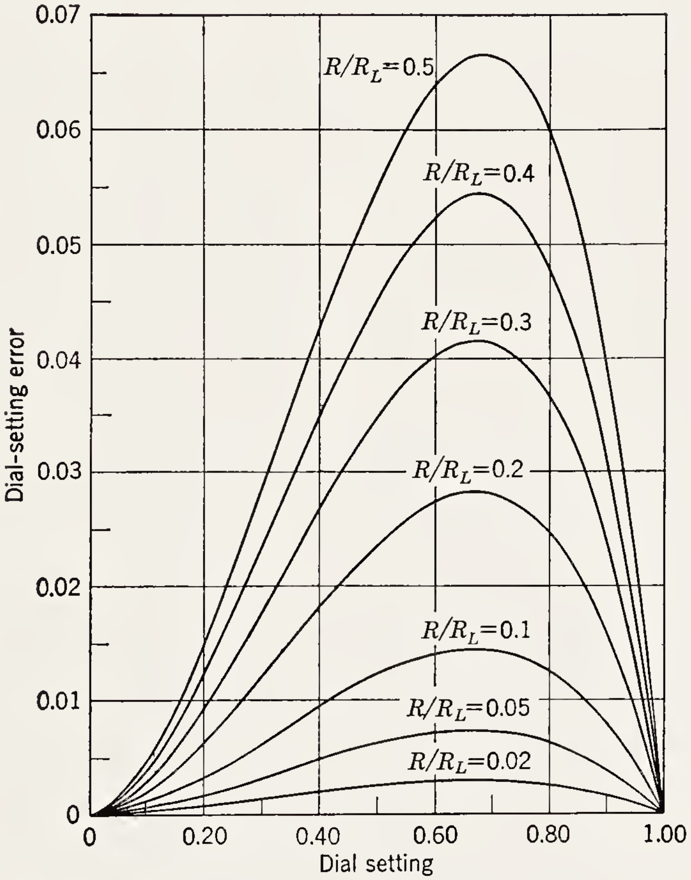

FIG. 1-6. Dial-setting error due to load resistance R_L.

It is evident that small values of resistances for potentiometers and large values of load resistances are desirable. The resistances of potentiometers used in d-c analog computers vary between 10,000 and 300,000 ohms. The lower limit of the potentiometer resistance is determined by the power of the amplifier driving the potentiometer and by the required resolution. The upper limit is fixed by loading considerations and by availability of wire-wound resistance elements. The load resistances which a potentiometer must drive vary between 10,000 ohms and 10 megohms. Therefore, since the potentiometer loading effect is proportional to R/R_L, the dial reading a may be in error by as much as 100 per cent or more in many cases.

Modern analog computers are so arranged that, when the operator sets up his problem, the potentiometer loading effect is easily ascertained. This is done by loading the potentiometer to be used in the problem in the same manner as in the problem and then applying a 100-volt signal to the potentiometer and adjusting the output until the desired fraction, a, is obtained.

1-4. THE SERVOMULTIPLIER

Figure 1-7 shows a schematic diagram of a servomultiplier. It consists of a number of linear potentiometers grouped together and driven by a servomotor. The reference voltage $\pm e_R$ volts is connected across one of the potentiometers. When the wiper-arm voltage e_f is fed back and subtracted from the input voltage e_i, the resulting error signal $e = e_i - e_f$ is sent through a high-gain servoamplifier and applied to the servomotor. The motor then drives the wiper arm in the proper direction to reduce the error to zero, i.e., to make $e_f = e_i$. In this way, the wiper-arm position of all the grouped potentiometers is proportional to the voltage e_i. If $\pm e_1$, $\pm e_2$, . . . , $\pm e_n$, etc., are applied across each of the remaining potentiometers, it is apparent that the variable voltages at the arm of the respective potentiometers will be $e_1 e_i/e_R$, $e_2 e_i/e_R$, . . . , $e_n e_i/e_R$, etc. Thus the servomultiplier can generate voltages proportional to the product of input voltage times the voltage across the multiplying potentiometers. Most practical servomultipliers contain five or six potentiometers. With one of the potentiometers used as the follow-up potentiometer, this leaves four or five potentiometers for multiplication. In practice the reference voltage $\pm e_R$ is generally ± 100 volts. This gives a constant of proportionality of $1/100$, so that the output voltage e_{on} from the nth-potentiometer wiper arm is $e_{on} = e_i e_n/100$.

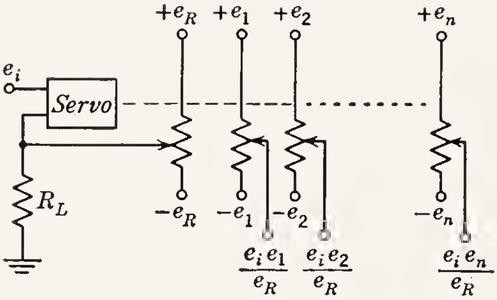

FIG. 1-7. Schematic diagram of the servomultiplier.

Referring again to Fig. 1-7, the rectangle labeled "servo" is used to represent schematically the servoamplifier and -motor. The resistance R_L is drawn separately to emphasize the loading effect which exists on the multiplying potentiometers, due to feeding the output voltages $e_ie_n/e_R = e_{on}$ as input voltages to operational amplifiers. Thus the multiplying potentiometers are loaded, as explained in Sec. 1-3. As a result the voltages obtained at the wiper arm deviate from the desired product. To overcome this effect, the resistance R_L is introduced to load the follow-up potentiometer in the same manner as the loading existing on the multiplying potentiometers. If the loading of all the multiplying potentiometers and the follow-up potentiometer is the same, the error in multiplication due to potentiometer loading will be automatically canceled. It is also common practice to center-tap all the potentiometers. Thus, if $\pm e_R$ is applied to the feedback potentiometer and $\pm e_1$, $\pm e_2$, . . . , etc., to the multiplying potentiometers, and if the center tap of all the potentiometers is grounded, we are assured of having the mechanical zero of the servomultiplier coincide with the electrical zero.

It has become standard terminology to call the top of the multiplying potentiometer + (plus) and the bottom − (minus). If this convention is observed, and a voltage $+e_1$ is placed on the top and $-e_1$ is placed on the bottom of the multiplying potentiometer, the wiper arm will read $+e_1e_i/100$. If the polarities are reversed, the wiper arm will read $-e_1e_i/100$. If the input voltage is either always + or always −, then only the top or the bottom of the multiplying potentiometer is required. If the voltage $+e_1$ is put on both the top and bottom of a multiplying potentiometer, there is obtained $|e_i|e_1/100$. In the reverse case, if $-e_1$ is used, there is obtained $-|e_i|e_1/100$.

In using the servomultiplier, it may not always be possible to program a given problem so that the same load is applied to each of the multiplying potentiometers. It is possible that a compromise has to be made whereby a servomultiplier is used with incorrect loading on one or more of the products. To investigate the magnitude of the errors involved when the multiplying potentiometer load is different from the follow-up potentiometer load, we consider again Eq. (1-17) for the effect of a load resistance on a potentiometer. Let R_{L_2} be the resistance used to load the follow-up potentiometer and R_{L_1} the load resistance seen by the multiplying potentiometer. To find the percentage error involved in multiplication due to $R_{L_1} \neq R_{L_2}$, we take the ratio of Eq. (1-17) for $R_L = R_{L_1}$ to Eq. (1-17) for $R_L = R_{L_2}$ to obtain

$$\phi = \frac{1 + a(1 - a)R/R_{L_2}}{1 + a(1 - a)R/R_{L_1}} \tag{1-19}$$

To find the point of maximum percentage error, we differentiate Eq.

(1-19) with respect to a and equate to zero to get

$$(1 - 2a)\left(\frac{R}{R_{L_1}} - \frac{R}{R_{L_2}}\right) = 0 \tag{1-20}$$

The condition $a = \frac{1}{2}$ is the point of maximum error which occurs when the wiper arm is at a point equal to one-half its travel. In applying this criteria to the servomultiplier, it is necessary to remember that the multiplying potentiometers are usually center-tapped and the center tap grounded. Thus, the point $a = \frac{1}{2}$ corresponds to one-half the travel of the servomultiplier either positive or negative. If the reference voltage is ±100 volts, then the maximum error will occur when the input voltage is ±50 volts. Using the value $a = \frac{1}{2}$, we obtain the equation for maximum percentage error as

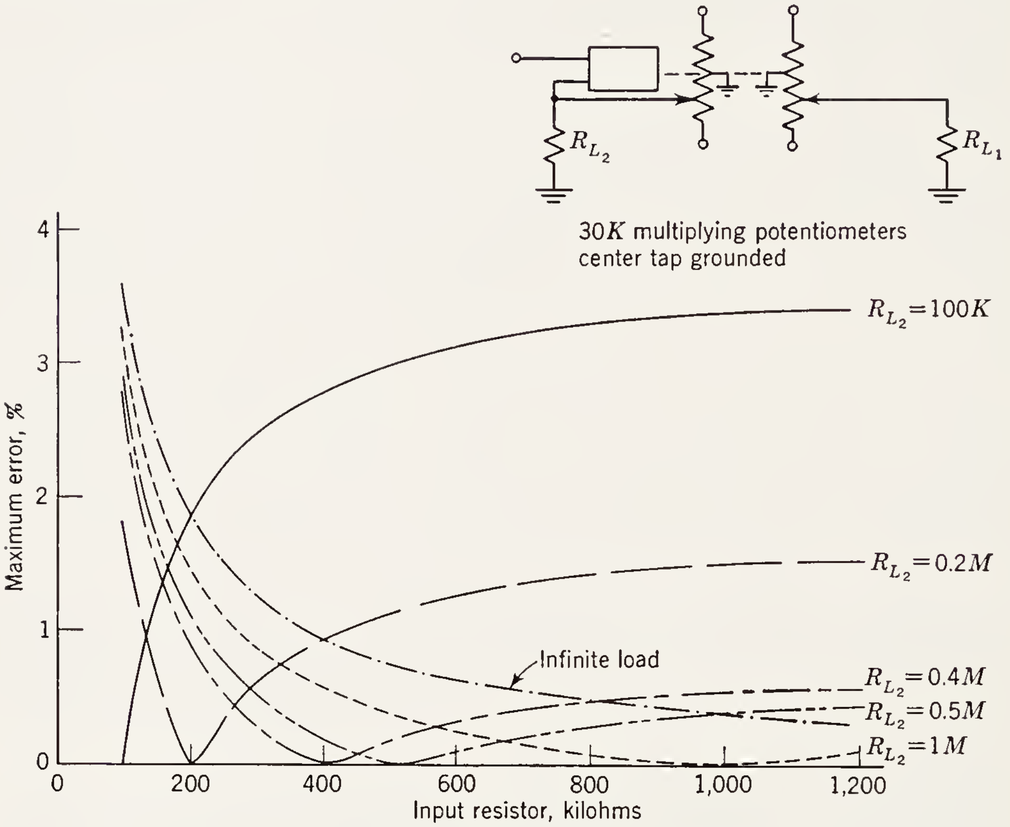

FIG. 1-8. Percentage error in multiplication due to incorrect loading.

$$\phi_{\max} = \frac{4 + R/R_{L_2}}{4 + R/R_{L_1}} \tag{1-21}$$

It then follows that the equation for the per cent maximum absolute error is

$$\%\ \text{error} = 100\left(1 - \frac{4 + R/R_{L_2}}{4 + R/R_{L_1}}\right) \tag{1-22}$$

A common value for the total resistance of servomultipliers is 30,000 ohms. Thus, if the servomultiplier is used with the center tap grounded, the value of R to be used in Eq. (1-22) is 15,000 ohms. Figure 1-8 shows a plot of the percentage error for various loadings of the multiplier potentiometers. From this figure it will be noticed that, in many cases, the absolute maximum error involved due to incorrect loading is less than 1 per cent for large changes in load on either side of the nominal. This fact can usually be exploited to give satisfactory results when it is not practical to load all the multiplying potentiometers identically. In the case where the load on one of the multiplying potentiometers, R_1, is greater than the follow-up potentiometer load R_L, a simple artifice illustrated in Fig. 1-9 can be used. A resistance R_2 is placed from the wiper arm to ground. The magnitude of R_2 is selected so that

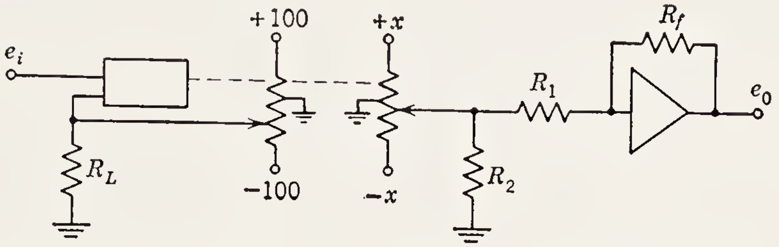

FIG. 1-9. Compensating for the potentiometer load when $R_1 > R_L$.

$$R_2 = \frac{R_1 R_L}{R_1 - R_L} \tag{1-23}$$

In this case the compensation is exact.

From what has been said, it would appear that multiplication presents no problem. This is not entirely the case, however, since the servomechanisms used in servomultipliers have a very limited bandwidth of frequency to which they will respond. Older-model servomultipliers are usually limited to something under 5 cps, while newer models have bandwidths which, in some cases, extend to 20 cps. This limitation means that the operator must evaluate each problem in terms of whether or not a servomultiplier is applicable. If not, it is necessary either to make a time transformation, i.e., to operate in slow time (discussed in Sec. 2-8), or, if this is not possible, to go to some means of high-speed multiplication. To determine whether or not a given problem exceeds the frequency-response capabilities of the servomultipliers, it has become customary to perform what is known as a time-scale check at the beginning of a given problem. This technique is explained in detail in Chap. 8.

1-5. THE PULSE-HEIGHT PULSE-WIDTH MULTIPLIER

In an effort to overcome the frequency-response limitations associated with the electromechanical servomultiplier, several attempts have been made at the development of an all-electronic multiplier. The most popular of these is the pulse-height pulse-width, or time-division, multiplier.

The time-division multiplier makes use of a square wave in which the ratio of the width of the positive pulse to the width of the negative pulse is varied in accordance with one input voltage and the amplitude of the wave is made proportional to the other voltage. The phasing of the wave is also determined by the second voltage.

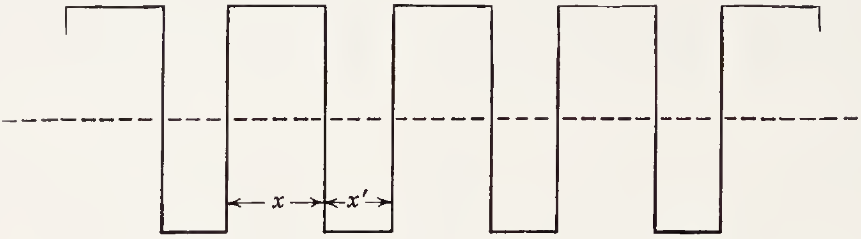

FIG. 1-10. Square wave used in time-division multiplier.

Figure 1-10 illustrates such a wave. Let x be the width of the positive pulse and x' the width of the negative pulse. If the input voltages are e_x and e_y, the x/x' ratio is adjusted so that

$$e_x = K_1 \frac{x - x'}{x + x'} \tag{1-24}$$

or

$$\frac{K_1 + e_x}{K_1 - e_x} = \frac{x}{x'}$$

where K_1 is a constant. The wave has positive and negative values equal to $+e_y$ and $-e_y$ so that during one complete cycle the area is xe_y while the voltage is positive and $x'e_y$ while it is negative. The difference in these two areas is proportional to the mean value. The output of the multiplier is obtained by smoothing the rectangular wave in an appropriate filter to obtain its average value. This gives a voltage

$$e_o = K_2(x - x')e_y \tag{1-25}$$

where K_2 is a constant. Thus, from Eqs. (1-25) and (1-24), we obtain

$$e_o = e_x e_y \frac{K_2}{K_1} (x + x') \tag{1-26}$$

where $x + x'$ is the periodic time for one cycle of the wave. Assuming that this is constant, $(K_2/K_1)(x + x')$ is also constant and can be set equal to K so that

$$e_o = Ke_xe_y \tag{1-27}$$

The constant K is usually adjusted to be equal to $1/100$, so that for $e_x = e_y = 100$ the product $Ke_xe_y = 100$, which does not exceed the normal operating limits of the computer. The repetition rate of the time-division multiplier is usually 10,000 to 15,000 repetitions per second.

An important feature of the time-division multiplier is that it can accept both positive and negative inputs for both e_x and e_y and gives an output voltage of the correct sign.

In a manner similar to the servomultiplier, it is possible to obtain the product of e_x and some other variable e_z simultaneously with the product e_xe_y. This is accomplished by using the basic square wave for which the ratio of the positive pulse to the negative has been determined by e_x and making the amplitude separately proportional to e_y and e_z. We thus obtain the products $e_xe_y/100$ and $e_xe_z/100$. This technique may extend to include up to approximately five simultaneous products.

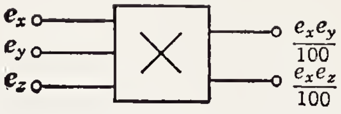

FIG. 1-11. Symbol for time-division multiplication.

Figure 1-11 shows the symbol commonly used to indicate the time-division multiplier. In this symbol the e_x input is referred to as the trigger and determines the quantity $x - x'$. The amplitude of the square wave is then modulated by the voltages e_y, e_z, etc.

1-6. ADDITIONAL METHODS FOR ELECTRONIC MULTIPLICATION

The servomultiplier and time-division multiplier are capable of accepting both + and − voltages for either e_x or e_y. Their output voltage e_o has the correct sign associated with the product $+e_xe_y$ or $-e_xe_y$. These multipliers are said to be capable of four-quadrant operation. Other types are capable of two-quadrant operation. Here the input voltage e_x may be either positive or negative, because the output voltage e_o will change sign with the input voltage e_x. The input voltage e_y, however, can be only either positive or negative throughout the computation. Such multipliers are called two-quadrant multipliers. Similarly a one-quadrant multiplier would be capable of accepting only voltages of one sign for either e_x or e_y.

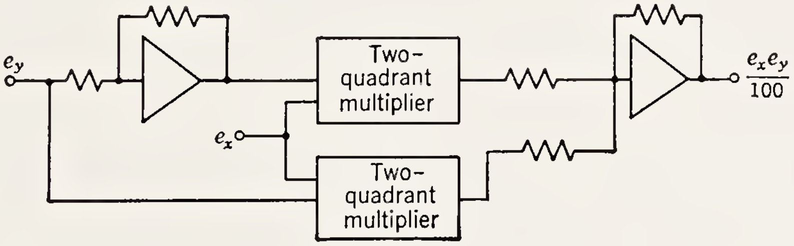

FIG. 1-12. The combination of two-quadrant multipliers to perform four-quadrant multiplication.

Generally speaking, four-quadrant multipliers can be built by combining the simpler one- and two-quadrant multiplying devices. Figure 1-12 shows how a two-quadrant multiplier can be made to serve for four-quadrant multiplication. In this figure it is assumed that the two-quadrant multiplier is a device which gives an output voltage as follows:

$$e_o = \begin{cases} \dfrac{e_x e_y}{100} & \text{for } e_y > 0 \\ 0 & \text{for } e_y < 0 \end{cases} \tag{1-28}$$

and no restriction on e_x. In applying two of these devices to achieve four-quadrant multiplication, for $e_y > 0$, the upper two-quadrant multiplier contributes 0 to the result, while the lower one contributes $e_x e_y/100$; for $e_y < 0$, the reverse situation prevails. Thus the output voltage is $e_x e_y/100$ regardless of the sign of e_y. The exact form of the circuit of Fig. 1-12 will vary with the characteristics of the two-quadrant multiplier used.

The multiplication of two voltages may be accomplished by taking advantage of the mathematical properties of logarithms. Thus consider the equation

$$e_1 e_2 = a^{\log_a e_1 + \log_a e_2} = \text{antilog}\,(\log_a e_1 + \log_a e_2) \tag{1-29}$$

Figure 1-13 shows a simple circuit for this equation, which consists of an amplifier summing two voltages which are proportional to the logarithms of the voltages e_x and e_y, respectively. Following the amplifier is an antilogarithm unit and amplifier which generates the desired end voltage. The over-all gain of the circuit is adjusted so that the product $e_x e_y/100$ is obtained. Devices for generation of the logarithm and antilogarithm of voltages can be constructed by using the principles used in constructing arbitrary-function generators. This subject is discussed in detail in Chap. 7.

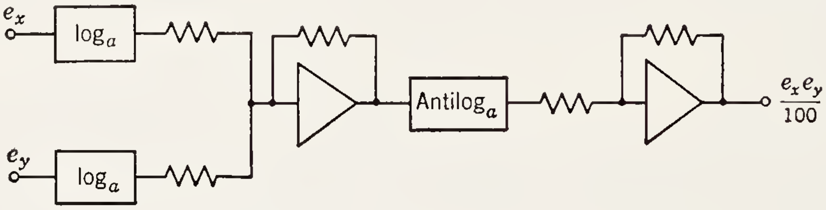

FIG. 1-13. Multiplication by the use of logarithms.

Since the logarithmic function is not defined for negative values of the argument, the application of this circuit is restricted to positive values of e_x and e_y (one-quadrant operation).

A multiplier can be constructed based on the relation

$$e_1 e_2 = \tfrac{1}{4}[(e_1 + e_2)^2 - (e_1 - e_2)^2] \tag{1-30}$$

Multipliers based on the use of Eq. (1-30) are called quarter-square multipliers. If squaring devices permitting negative as well as positive input voltages are available, a quarter-square multiplier will permit direct multiplication of both positive and negative variables (four-quadrant operation). Chapter 7 discusses the generation of special arbitrary functions, among which are the squaring function. The application of these devices to the performing of four-quadrant multiplication is given there.

1-7. THE SERVO RESOLVER

The servo resolver is a special kind of servomultiplier. In the servo resolver a precision sine-cosine potentiometer, which will deliver output voltages proportional to the sine or cosine of the brush angle, is used in the place of a multiplying potentiometer of the servomultiplier. Two of these potentiometers are mounted concentrically within each resolver and consist of resistance wire wound on a card which has a shape similar to a sine wave. The card is made to precise dimensions and accurately wound, since the accuracy of the sine function generated depends upon the shape of the card. The name resolver is derived from the action of this device as a converter of a vector displacement into its rectangular components.

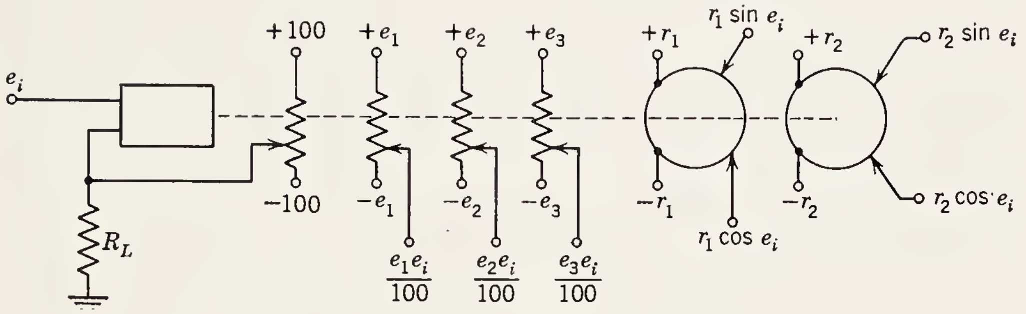

FIG. 1-14. Schematic diagram of a servo resolver.

The servomotor drives a pair of brushes displaced from each other by 90 deg. The sine is available from one arm and the cosine from the other. The operating range of the resolver is from -180 to $+180$ deg. In order not to exceed the ± 100-volt limits within which the computer works, the ratio of 2 deg/volt is used in resolvers. The basic operation carried out by resolvers is that of multiplying a variable voltage by the sine and cosine of the angle θ. Nonlinear potentiometers may be used in a manner very similar to linear multiplying potentiometers. In all applications of resolvers it is necessary to apply the correct loading, as no compensating techniques are applicable. By applying a voltage $+r$ to the high end of one of the resolving potentiometers and $-r$ to the low end, one may "resolve" a polar quantity $r(\theta)$ into its rectangular components $r \cos \theta$ and $r \sin \theta$. The mode of operation in which one obtains the rectangular components of a polar quantity is called "rectangular output." Figure 1-14 shows the schematic representation of the servo resolver. Frequently the two sine-cosine potentiometers are shown concentric to each other. As indicated in Fig. 1-14, the servo resolver usually contains three or four linear potentiometers in addition to the sine-cosine potentiometers. These potentiometers can be used to obtain the product of two voltages, in the same manner as the servomultiplier is used.

The servo resolver has a second mode of operation called the polar mode. In this mode of operation the servo resolver performs the mathematical operations

$$E = \sqrt{x^2 + y^2} \tag{1-31}$$

and

$$\tan\theta = \frac{y}{x} \tag{1-32}$$

This transformation is equivalent to knowing the rectangular coordinates of a point and transforming the coordinates to obtain the polar coordinates of the point. Figure 1-15 shows the circuit hookup of the resolver for accomplishing the rectangular- to polar-coordinate transformation.

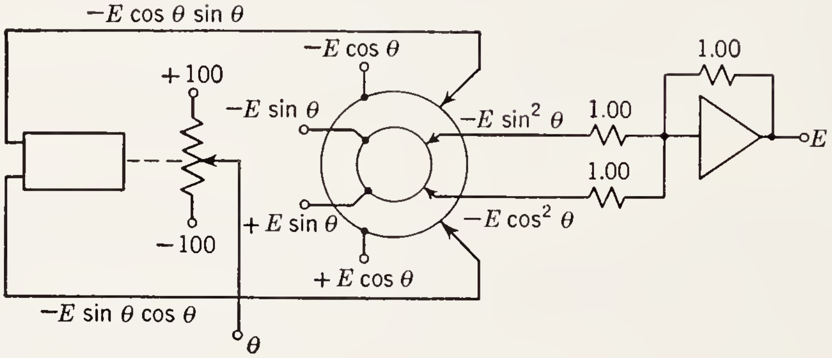

Fig. 1-15. Circuit for transforming from rectangular to polar coordinates.

The operation of this circuit may be explained as follows: The quantities $\pm x = \pm E\cos\theta$ and $\pm y = \pm E\sin\theta$ are available in the computer, and it is required to compute the quantities E and θ from these relations. To accomplish this, we multiply $\pm x$ by $\cos\theta$ and $\pm y$ by $\sin\theta$ to obtain $\pm x\cos\theta = \pm E\cos^2\theta$ and $\pm y = \pm E\sin^2\theta$. If we sum these two quantities, there is obtained

$$\pm x \pm y = \pm E(\cos^2\theta + \sin^2\theta) = \pm E \tag{1-33}$$

To obtain the angle θ, the quantities $-E\cos\theta\sin\phi$ and $-E\sin\theta\cos\phi$ are fed back to the servomotor. The servomotor will rotate to a position such that

$$-E\cos\theta\sin\phi = -E\sin\theta\cos\phi$$

or

$$\tan\phi = \tan\theta$$

and

$$\phi = \theta \tag{1-34}$$

Thus the voltage off the follow-up potentiometer is proportional to the angle θ. The angle θ is usually scaled to give ½ volt/deg of θ.

1-8. OUTPUT DEVICES OR RECORDERS

The solutions to problems solved on the analog computer appear as voltage variations in the output of various operational amplifiers throughout the computer. The output devices used to record these voltage variations are essentially recording voltmeters. These devices automatically plot graphs of one or more amplifier output voltages against the time t as the independent variable. In addition some output devices will plot one computer variable versus another.

The most common recording device found in an analog-computer laboratory consists of conventional six-channel direct-writing magnetic recorders of the general type shown in Fig. 1-16. This type of recorder consists of a paper-drive mechanism and six pen motors. Each pen motor is essentially a D'Arsonval-meter movement with a lightweight writing pen attached to it. The pens are hollow, and the ink is fed through them to ink the recording paper. Another version of essentially the same type recorder uses tiny heated wires which produce black traces on heat-sensitive paper. The direct-inking devices usually produce what is known as curvilinear recording in that the data at any instant lie on the arc of a circle with the pivot point of the pen as center. The heat-sensitive devices usually produce rectilinear recordings.

The more recent six-channel recorders contain some rather elaborate calibration and computer control features. Along this line the recorder control panel contains all the standard computer operating controls required for operating the computer from the recorder. This means that operation of the entire computer installation may be controlled from the recorder, which, of course, is where the operator will generally be standing. The operator, standing at the recorder, can thus begin a run and watch the results as the run progresses, having the problem under control at all times, without having to move back and forth between computer and recorder. The automatic-calibration feature usually consists of a system which records the pen zero position, volts/unit deflection, and paper speed at the beginning or end of a computer run.

Direct-inking magnetic recorders are capable of accuracies of 1 to 5 per cent. The frequency response of most of these units is flat up to 100 cps.

Perhaps the next most common output-recording device is the servo-driven plotting board. In these devices a servomechanism loop similar to that used in the servomultiplier is used for positioning the x and y coordinates of a plotting pen. The recording surface usually consists of

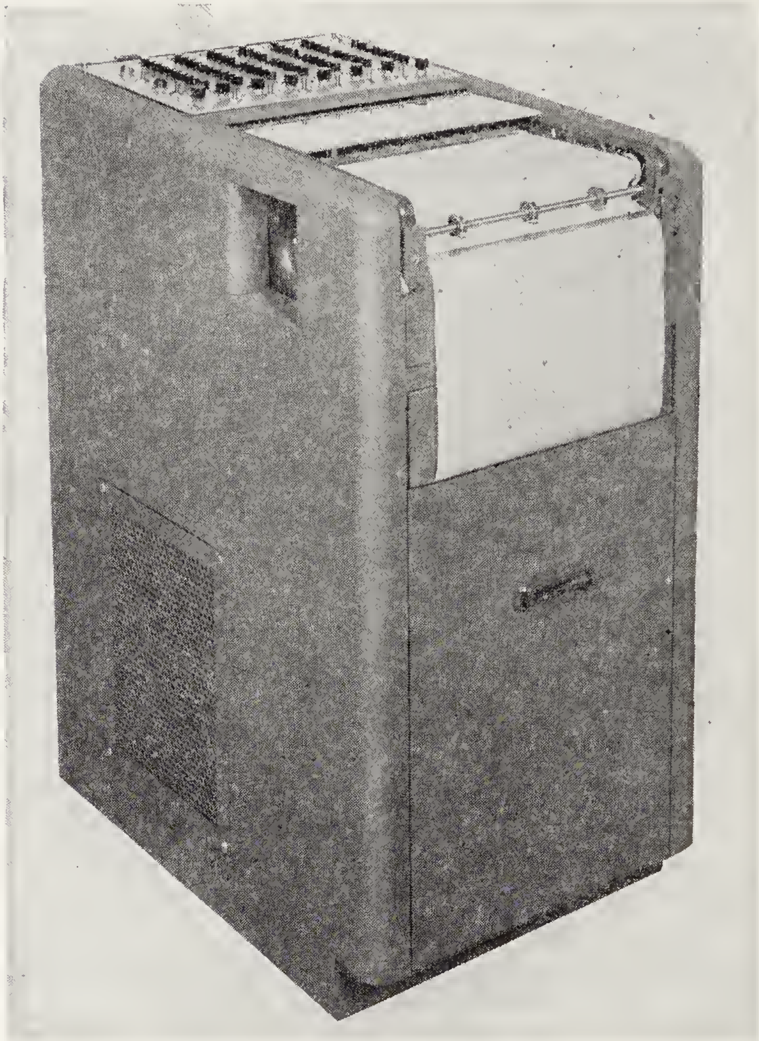

FIG. 1-16. PACE six-channel recorder, model 1902C. (*Courtesy Electronics Associates, Inc.*)

a flat surface. The size of the surface varies, but of the smaller models the 11- by 17-in. size, to accommodate the standard 11- by 17-in. graph paper, has become very popular. Larger models range up to 30 by 30 in. Figure 1-17 shows the model 1100, 11- by 17-in. recorder manufactured by Electronics Associates. In this recorder the parallel bar, or carriage, running across the recording surface is positioned along the long axes of the recorder by the x servomotor. The writing pen is attached to the x position bar and moves along the short axes of the recorder as determined by the y input driving the y servomotor. Since the xy coordinates on the plotting surface are determined by two independent servomechanisms, this device can be used to plot one voltage with respect to another voltage.

To facilitate the insertion and removal of paper, a vacuum hold-down system is employed. The recording accuracy varies with plotting speed. Typical figures are: 0.1 per cent for plotting speeds up to 10 in./sec; 0.2 per cent for plotting speeds up to 15 in./sec. The frequency response is best stated in terms of phase shift; typical specifications are: less than 5 deg phase shift for plotting a double amplitude of 8 in. at 1 cps; less than 5 deg phase shift for plotting a double amplitude of 1 in. at 2 cps. The acceleration limits of pen and carriage are approximately 750 in./sec^2 and 250 in./sec^2, respectively.

During the setup process involved in the solution of a problem it is necessary to set the potentiometers so as to obtain an output voltage which is a prescribed fraction of the input voltage. The integrators must have the correct initial conditions introduced on them. To accomplish these items conveniently, the outputs of the amplifiers and the wiper arms of the potentiometers are all brought to a central point and placed on a multiposition switch. A voltmeter can be placed on the arm of this switch, and, by scanning all the positions, the voltages out of all the amplifiers and wiper arms of the potentiometers can be read. Four types

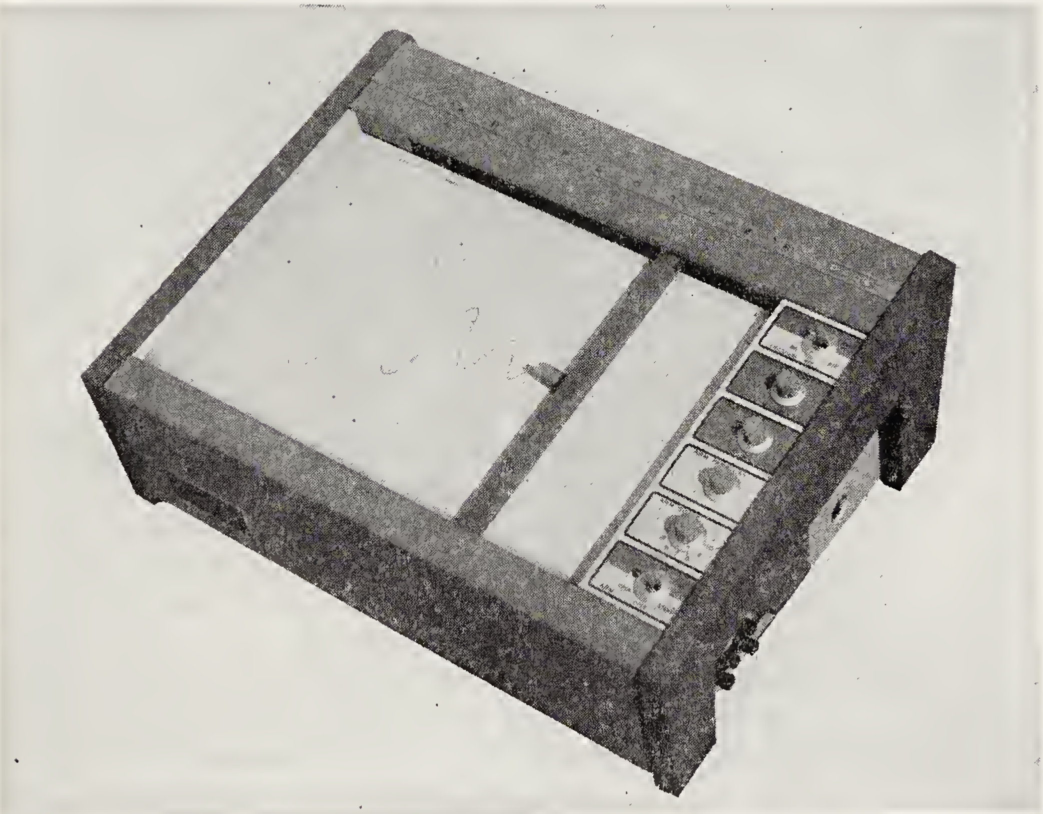

FIG. 1-17. PACE servo plotting board, model 1100E. (*Courtesy Electronics Associates, Inc.*)

of voltmeters are used for this purpose. These are the conventional high-impedance vacuum-tube voltmeter, the servo voltmeter, the null indicator, and the digital voltmeter. The vacuum-tube voltmeter cannot be relied upon where accurate results are required; its primary usefulness lies in giving an order of magnitude. Figure 1-18 shows a servo voltmeter. This unit provides voltage measurements in the range of ±100 volts. Answers are presented in terms of percentages of reference voltage on a direct-reading three-digit counter. For fast, accurate reading the value is presented to three places by figures, with a fourth place indicated by 0.02-volt graduations. The servo voltmeter can read voltages accurately to 0.025 per cent of the operating range, which is 200 volts.

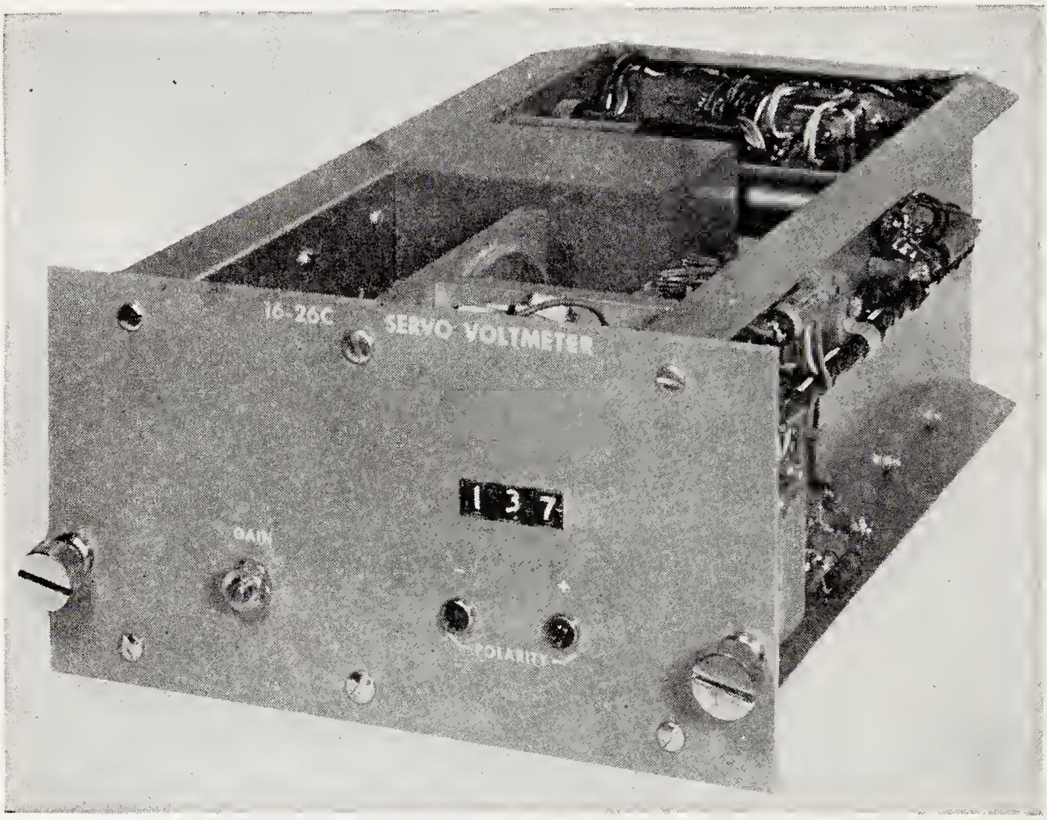

FIG. 1-18. PACE servo voltmeter, model 16-26C. (*Courtesy Electronics Associates, Inc.*)

The null indicator is a refinement of the vacuum-tube voltmeter which allows the voltage to be measured to be compared with a precisely known voltage. The null-indicator circuit shown in Fig. 1-19 employs a converter to produce an alternating signal when any voltage is present across the terminals of the d-c meter. This signal is amplified through the first triode, rectified, and applied to the capacitor in the grid circuit of the second triode. The charge across this capacitor keeps this triode cut off as long as a voltage appears across the d-c meter terminals. When the voltage across the meter terminals falls to zero, the capacitor in the grid circuit of the second triode will discharge through the rectifiers and the tube will conduct, causing the neon indicator to light. If the voltage across the meter is made to be the difference between the voltage to be set and a known calibrated voltage, then lighting of the null indicator indicates the set value. The null indicator has been successfully applied in the GEDA L3 computer. On this computer voltages can be read with fair accuracy to within ¼ volt.

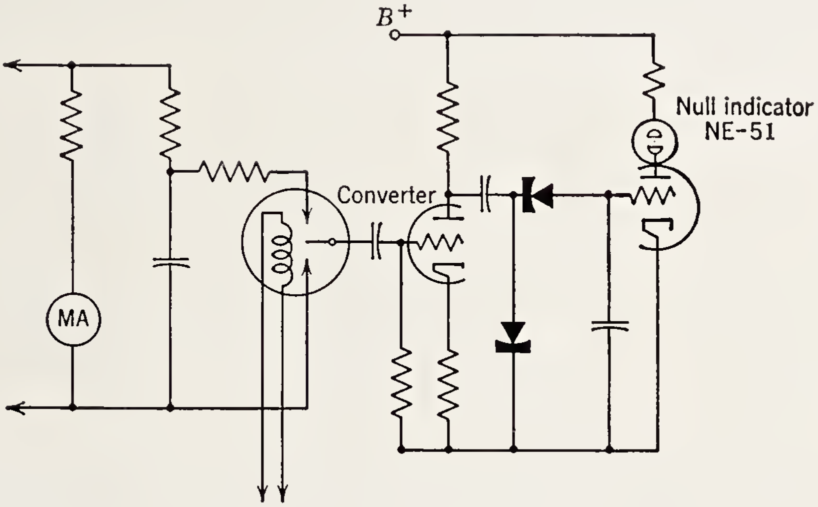

Fig. 1-19. Null-indicator circuit.

The digital voltmeter is a common output device which provides a numerical display of a voltage measurement. It is an automatic potentiometer which produces a feedback voltage proportional to the unknown input. The answer is displayed on a digital read-out panel. The difference between the feedback and input voltages causes a digital positioning system to adjust a voltage divider to reduce the difference, or "error voltage," to zero. When this condition is achieved, the voltmeter is said to be balanced and the unknown voltage will be indicated on the visual read-out. The voltage divider is actually a series of accurate, fixed resistors arranged to form a bridge, whose configuration is composed of succeeding decades connected across one section of each preceding decade. The setting of the voltage divider is controlled by stepping switches in the digital positioning system. In this manner, when an entire decade has been traversed by the switches, its resistance may be represented by one step of the next higher decade, in much the same fashion as a mechanical counter. It is not the purpose of this text to discuss the complete theory of analog to digital conversion; therefore we shall content ourselves with some schematic diagrams which illustrate the operation of such devices.

Figure 1-20 is a schematic diagram of a digital voltmeter being used to read the voltage at the wiper arm of a potentiometer.

The circuit shown in Fig. 1-21 indicates the method for a four-digit voltmeter in which the feedback voltage from the digital voltage divider is produced. In operation, owing to the switching logic of the stepping switches in the voltage comparator, the number 4 decade divider will first choose the feedback voltage closest to the input voltage within its range. Following this, the number 3 decade, with the output of the number 4 decade across it, will then select the tap corresponding to the input voltage within its range. This continues until all the digits of the value have been selected. If the polarity is improper, the decade 4 switch will run through its cycle and cause the polarity switch to operate.

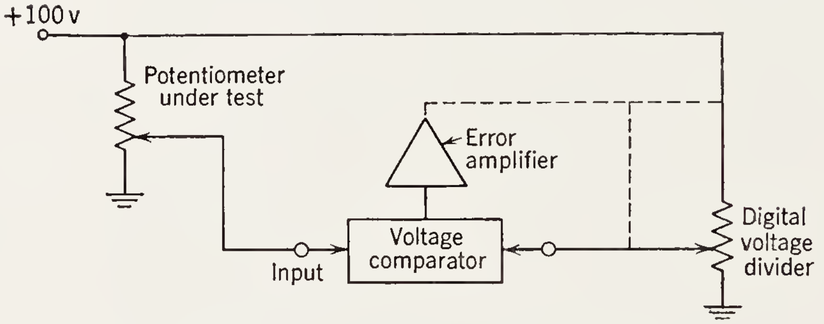

Fig. 1-20. Schematic diagram of digital voltmeter.

The familiar cathode-ray oscilloscope should be listed among the output devices for the analog computer. Because of its high frequency response it is particularly useful for the investigation of computer instability. In this case high-frequency oscillations may occur, which would not be detected by the output devices previously described. The oscilloscope proves to be an exceedingly useful device for each investigation. Another application of the computer in which the oscilloscope is particularly useful is repetitive operation. In this mode of operation the computer is made to scan the problem solution at a relatively high rate. By a suitable switching arrangement the solution to the problem is repeated continually. If an oscilloscope whose screen persistence is long enough is used as the output device, the solution to a problem is presented as a continuous curve on the oscilloscope. Polaroid cameras are available by which permanent records can be made.

1-9. PROBLEM BOARDS

The solution of a differential equation by the use of the analog computer involves the interconnection of devices capable of addition, subtraction, multiplication, integration, etc. In general more than one of

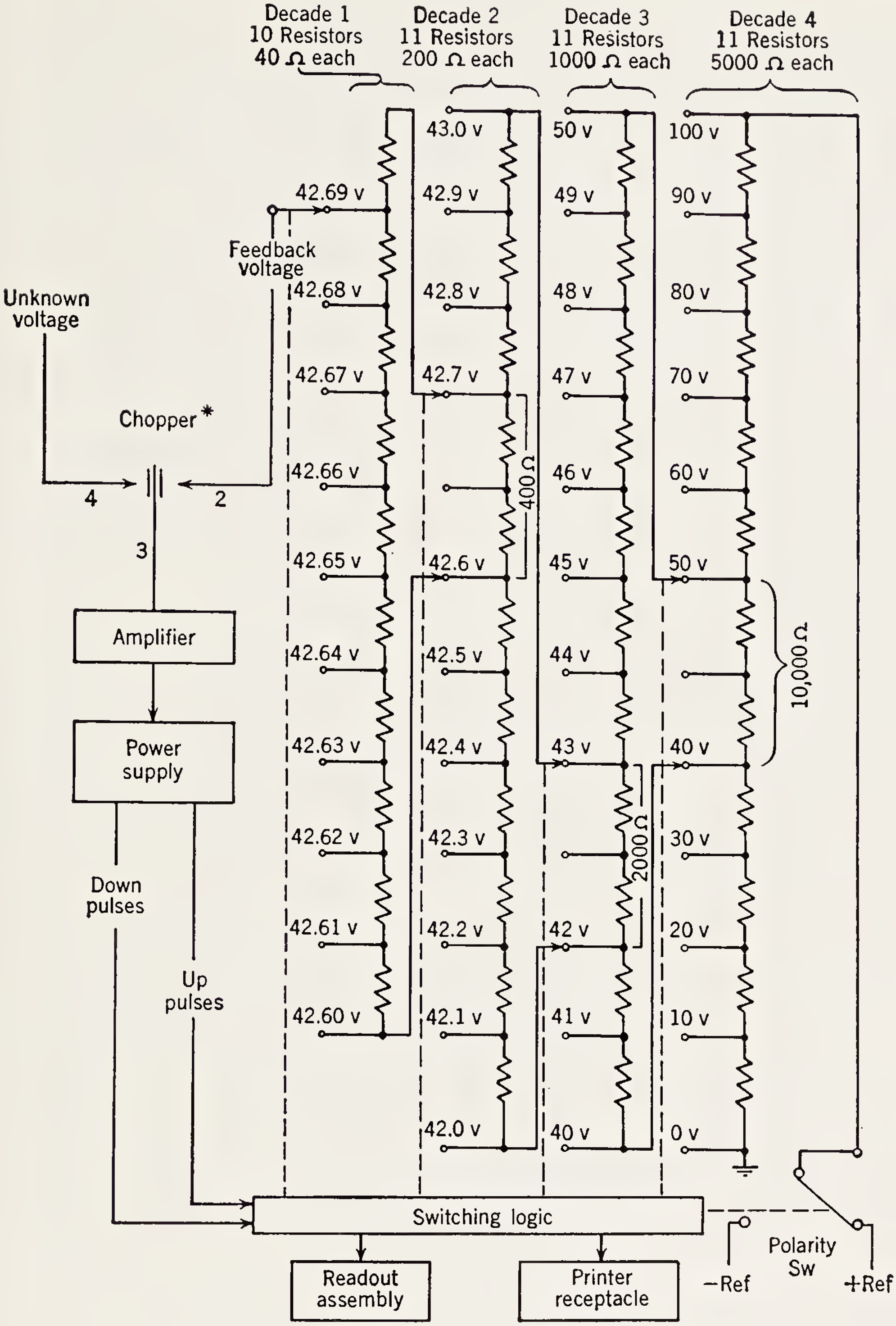

FIG. 1-21. Digital-voltmeter voltage-divider functional diagram.

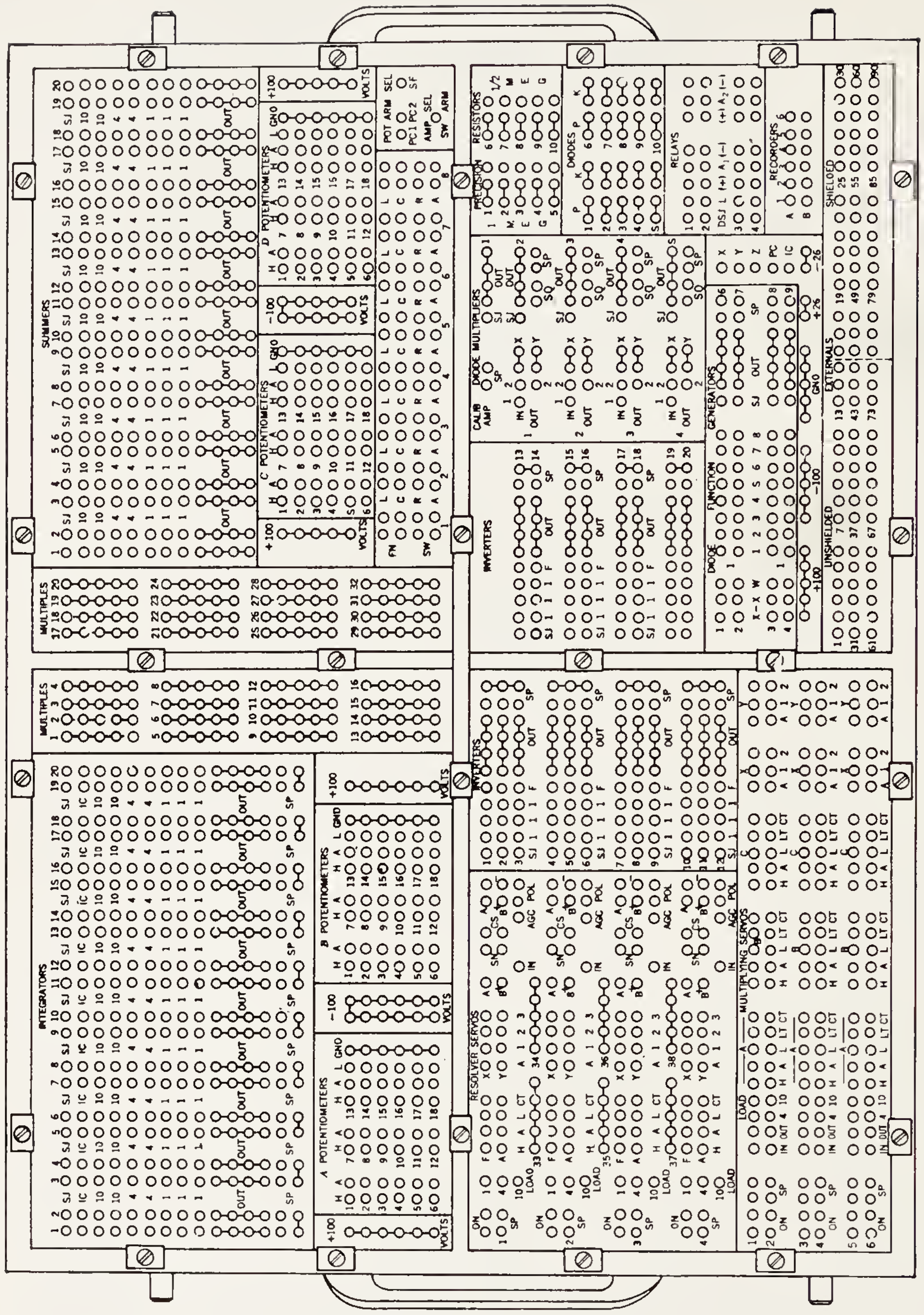

FIG. 1-22. Typical prepatch problem board. *(Courtesy Reeves Instrument Company.)*

these operations will be involved, and their occurrence will be in an arbitrary manner. To provide a flexible system which is capable of accommodating any arbitrary arrangement of operations, a removable-patch-board system has been devised. Patch boards consist of a grouping of the inputs and outputs of 20 to 80 amplifiers, the tops and arms of 30 to 80 potentiometers, and associated auxiliary equipment such as multipliers and function generators. Figure 1-22 shows a typical patch-board layout for the REAC A400 series computer. This particular patch-board layout makes provision for 60 amplifiers, of which 20 are wired as integrators, 20 as summing amplifiers, and 20 as summing inverters. In the case of the integrators and summing amplifiers 7 input resistors are provided. Of these, 3 have a resistance of 1 megohm, 2 have a resistance of 0.250 megohm, and 2 have a resistance of 0.100 megohm. The hole marked IC on the integrators is for the purpose of introducing the initial conditions. Each amplifier has 4 common outputs. The feedback resistor, whose value is 1 megohm, is permanently wired in the case of the summing amplifiers. The integrating amplifiers have their feedback completed by means of plugging in a wire from the output to the hole directly below the amplifier outputs. The inverting amplifiers have 3 input resistors all equal to 1 megohm and a 1-megohm feedback resistor. The feedback resistor for the inverting amplifiers is connected in the same manner as the feedback capacitor in the case of the integrators. The hole marked SJ on all the amplifiers is the summing junction or input grid of the amplifier. Seventy-two potentiometers are shown on this patch board in 4 groups of 18. The first 12 potentiometers of each group have only 2 terminals. These have the top marked H and arm marked A. The bottom of each of these potentiometers is grounded. The last 6 potentiometers have 3 terminals. In this case the bottom terminal as well as the top and wiper arm is brought out to the patch board. Also contained on the patch board are 6 servomultipliers and 4 resolver servos. Each of the servomultipliers has 5 multiplying potentiometers. The multiplying potentiometers are set up to have a normal load of 1 megohm. Provisions are made by means of the holes marked 4 and 10 whereby loads of 0.250 and 0.100 megohm can be applied. The resolving servos have 4 multiplying potentiometers and 2 sine-cosine potentiometers. Four multipliers of the quarter-square type are also available on this board. The remaining equipment consists of 4 arbitrary-diode-function generators, plug-in resistors, diodes, and relays. Ninety external connections for interconnecting this computer to a similar unit are provided. Scattered throughout the board will be found the ±100-volt reference voltages. Ground is similarly distributed. The multiple connections are common jacks for the purpose of providing additional connections where the requirement exceeds the holes provided.

PROBLEMS

1-1. Determine the output of the circuit of Fig. P1-1.

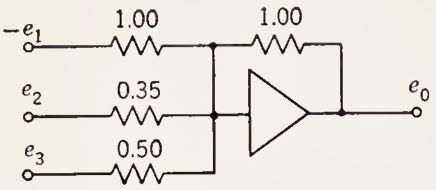

FIG. P1-1.

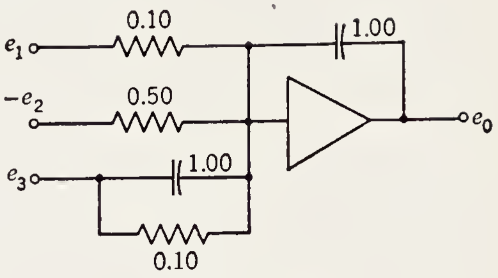

FIG. P1-2.

1-2. Determine the output of the amplifier of Fig. P1-2.

1-3. Determine the output of the amplifier of Fig. P1-3, where a is the dial setting of the potentiometer.

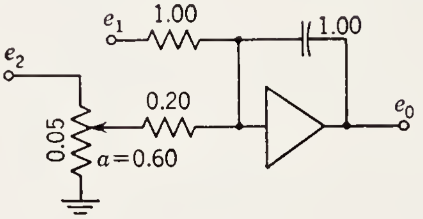

FIG. P1-3.

1-4. Using a 0.1-megohm potentiometer as a voltage divider (see Fig. 1-5) to obtain the voltage ratio 0.7, determine for what values of load resistance the dial setting will differ from 0.7 by more than 1 per cent (that is, 0.007 in dial reading).

1-5. Determine the output of the circuit of Fig. P1-5.

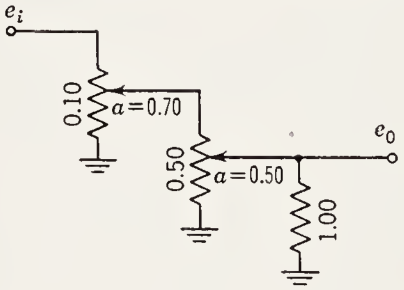

FIG. P1-5.

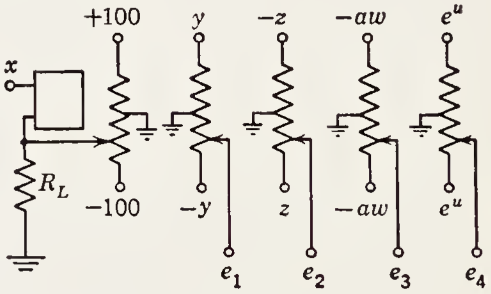

FIG. P1-6.

1-6. Given the servomultiplier of Fig. P1-6 with the inputs shown:

a. Determine the outputs.

b. If +25 volts is applied to the top of the follow-up potentiometer and −25 volts is applied to the bottom, what are the outputs?

c. If the input is $-x$, what are the outputs?

d. If e_1, e_2, and e_3 are loaded with 0.1 megohm and e_4 is loaded with 0.2 megohm, set up a circuit which compensates for this loading exactly.

e. If e_1 is loaded with 0.2 megohm, e_2 with 0.5 megohm, e_3 with 1 megohm, and e_4 with 10 megohms, determine R_L so that the error due to improper loading is less than 1 per cent. (Assume that the multiplying potentiometers are 30 kilohms.)

1-7. Determine the outputs of the servomultipliers of Fig. P1-7.

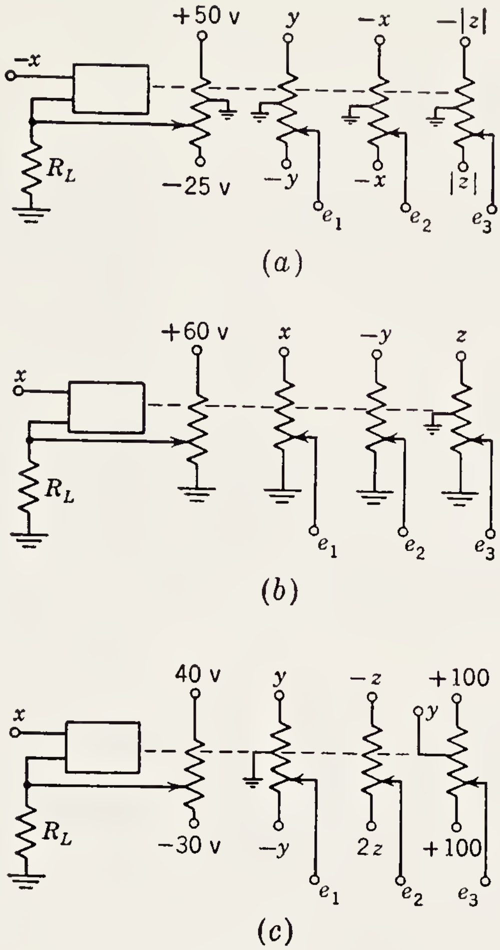

FIG. P1-7.

1-8. What are the outputs of the circuit of Fig. P1-8? What are the limitations on the values of y and x?

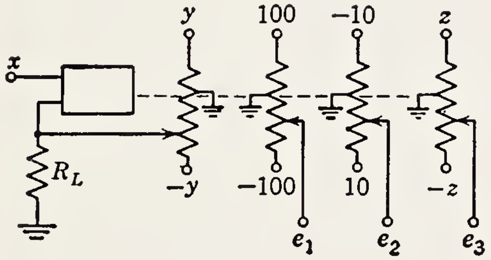

FIG. P1-8.

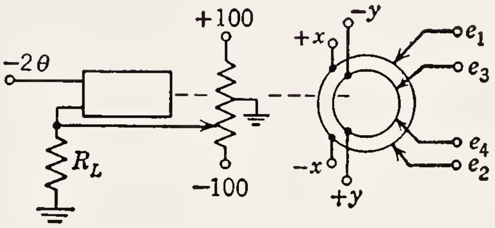

FIG. P1-10.

1-9. Given two devices that form the quantity $(x|x|)/100$, set up a quarter-square multiplier. Determine the outputs of this multiplier for inputs of all polarities; therefore determine the quadrants for which this can be used as a multiplier.

1-10. If the quantity θ is scaled to give 1 volt/deg, determine the outputs of the resolver of Fig. P1-10. (*Note:* This input is four times the normal input to a resolver.)

1-11. What are the outputs of the resolver of Fig. P1-11 if θ is scaled at 1 volt/deg?

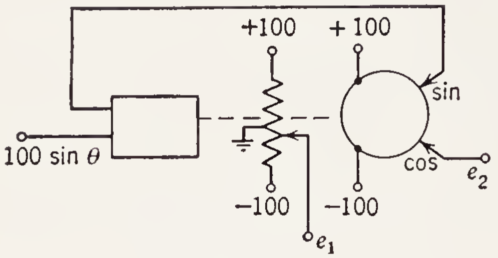

FIG. P1-11.

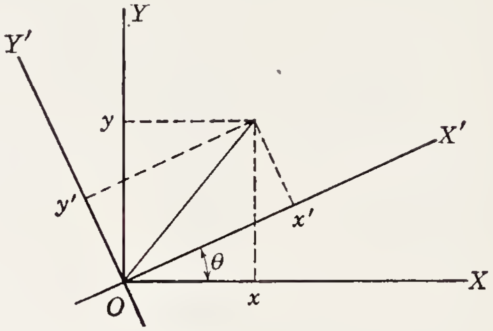

FIG. P1-12. Rotation of axes OX, OY to OX', OY'.

1-12. The rotation of one rectangular-coordinate axis OX, OY to another coordinate axis OX', OY' (Fig. P1-12) can be accomplished by means of the equations

$$\begin{aligned} X &= X' \cos \theta - Y' \sin \theta \\ Y &= X' \sin \theta + Y' \cos \theta \end{aligned}$$

Set up a circuit to perform this operation.

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1. Johnson, C. L.: "Analog Computer Techniques," McGraw-Hill Book Company, Inc., New York, 1956.

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4. Soroka, W. W.: "Analog Methods in Computation and Simulation," McGraw-Hill Book Company, Inc., New York, 1954.

5. Ragazzini, J. R., R. H. Randall, and F. A. Russel: Analysis of Problems in Dynamics by Electronic Circuits, *Proc. IRE*, May, 1947, pp. 444–452.

CHAPTER 2

Problem Planning and Programming

2-1. INTRODUCTION

In many respects the task of planning for and programming the analog computer may be looked upon as a bookkeeping procedure. As in keeping books, it is required that a systematic procedure be adopted. For this purpose a set of standard symbols for the various operations involved in analog computations has come into common usage. Two methods of diagramming problems have become quite common, and various work sheets have been developed.

In preparing a problem for the computer it is necessary to arrange the equations in a form which can be programmed. Scale factors must be selected to limit the problem variables to within the operating range of the computer and, if necessary, time transformations made to bring the response rates of the problem within the capability of the computer.

2-2. STANDARD SYMBOLS

A system of standardization of symbols for programming of analog computers has come into common usage. As with any system, variation of usage exists among users, but the system presented here appears to be fairly widely accepted in most computer laboratories.

In Chap. 1 the fundamental components of the analog computer were introduced, and a diagrammatic representation of these components was given as they were described. These representations are collected here for easy reference. Additional symbols are introduced to take care of interconnections between component parts and interconsole connections from one computer to another. In the interests of simplicity it has become common on many machines not to show the input and feedback resistors and capacitors but to replace the entire amplifier with one symbol which indicates the operation performed. The table given as Fig. 2-1 shows the equivalence between the two representations.

It is obvious that, in programming a large problem, the simplified symbol system has many advantages. Many problems, however, require fre-

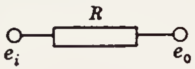
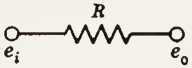
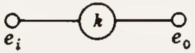
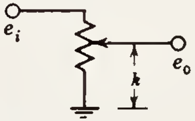
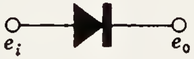
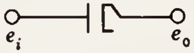
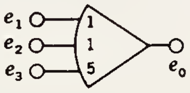
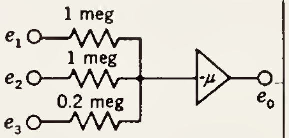
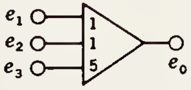
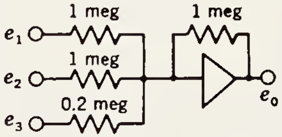
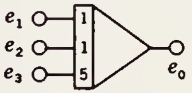
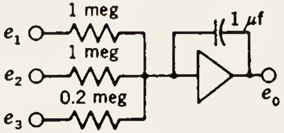
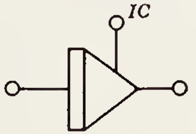
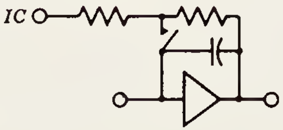
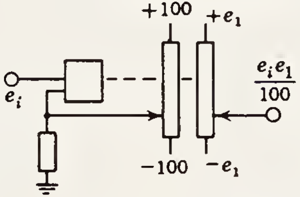
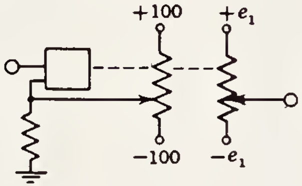
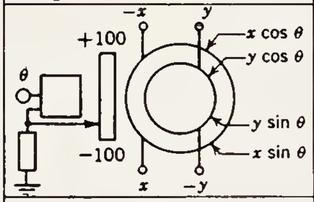
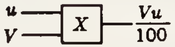
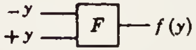
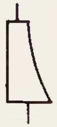
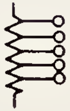
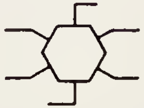
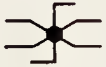
Symbol	Equivalent symbol	Operation performed
		Resistance $e_o - e_i = iR$
		Potentiometer $e_o = ke_i$ $0 \leqslant k \leqslant 1$
		Diode element
	 1 meg e_1 1 meg e_2 0.2 meg e_3 $-\mu$ e_o	High gain amplifier $e_o = -\mu(e_1+e_2+5e_3)$
	 1 meg 1 meg e_1 1 meg e_2 0.2 meg e_3 e_o	Summing amplifier $e_o = -(e_1+e_2+5e_3)$
	 1 meg 1 μf e_1 1 meg e_2 0.2 meg e_3 e_o	Summing integrator $e_o = -\frac{1}{s}(e_1+e_2+5e_3)$
		Initial condition
 +100 $+e_1$ e_i $\frac{e_i e_1}{100}$ −100 $-e_1$		Servo-multiplier
 $-x$ y +100 $x \cos\theta$ $y \cos\theta$ θ $y \sin\theta$ $x \sin\theta$ −100 x $-y$	Same	Servo resolver
	Same	Electronic multiplier
	Same	Arbitrary-function generator
		Tapped potentiometer
		Multiple

FIG. 2-1. Table of standard symbols.

quent changes of feedback and input components, particularly in using high-gain and implicit-function techniques. Here it is advantageous to have the components drawn out in detail. In the discussions to follow in this text, since the objective is to bring about an understanding of the particular circuits considered, the practice of drawing out the input and feedback resistors in detail will be followed.

To ease the burden associated with drawing out a complicated diagram for programming the wiring of a problem, several templates have been developed with the standard symbols. The most noteworthy of these is the one developed by Electronics Associates, Inc. This template is shown as Fig. 2-2.

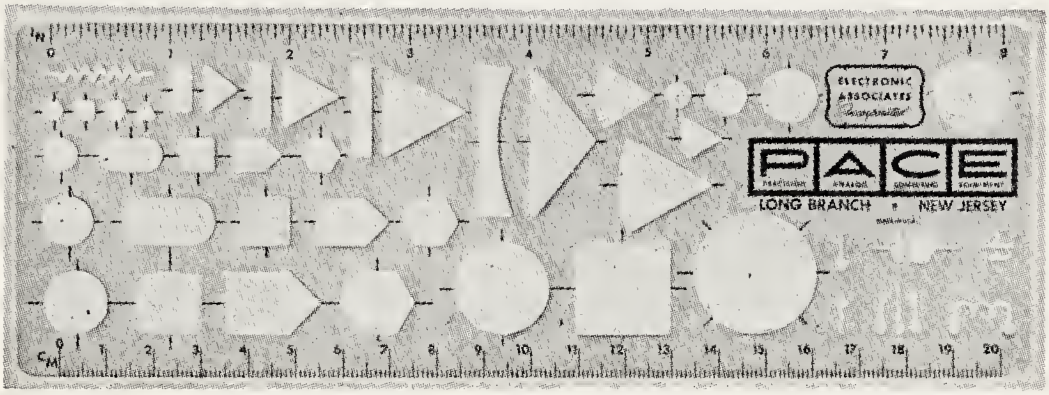


FIG. 2-2. PACE diagramming template. (*Courtesy Electronics Associates, Inc.*)

2-3. METHODS OF DIAGRAMMING

Two approaches to the detailed diagramming of the computer circuit wiring for a given problem have become common. These can be classified roughly as the "open style" and the "closed style" of diagramming.

In the open style of diagramming only the major components are drawn, with the various inputs to the components indicated; no interconnections are shown. This style is illustrated in Fig. 2-3. In this figure the numbered circles are potentiometers; they are indicated twice to show both input and output connections.

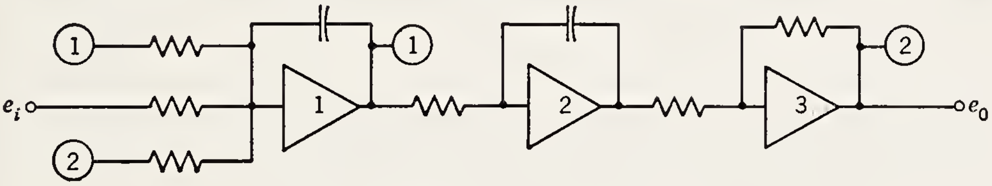


FIG. 2-3. Open-style circuit diagram.

In the closed style of diagramming all the major components and their interconnections are presented. It is in essence a complete wiring diagram of the problem and should have an exact one-to-one correspondence to the actual computer wiring. Figure 2-4 shows the closed-style diagram corresponding to the circuit of Fig. 2-3. It would appear to be more complicated than the open-style diagram. The closed style of diagramming, however, has one very distinct advantage over the open style, i.e., its one-to-one correspondence to the actual computer wiring. This is particularly advantageous with large problems which may involve 200 to 300 amplifiers, several hundred potentiometers, and proportionate amounts of associated equipment, located in several different computer consoles. In a problem of this magnitude, once the computer diagram has been established, the number of interconnecting wires may run into the thousands. It then becomes necessary to diagram the problem in a manner that allows for routine checking. The only practical way of achieving this is to have an exact one-to-one correspondence between the diagram and the computer wiring. This ensures that, once the circuit has been established and verified to be correct, the checking of the computer wiring against the diagram can be accomplished by a routine tracing and matching of wire for wire.

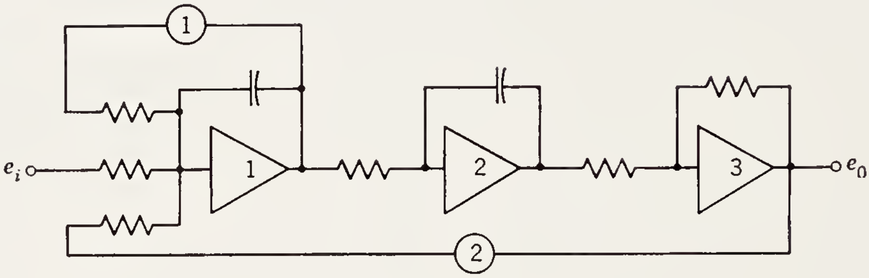


FIG. 2-4. Closed-style circuit diagram.

2-4. COMPUTER WORK SHEETS

A variety of computer work sheets has been developed. The work sheet shown as Fig. 2-5 has the advantage of being read like a road map. The major components are listed in a table and their location on the diagram given in terms of x and y coordinates of the paper. In practice, in working large problems, it has been found advantageous to prepare a separate diagram for each computer console used and to employ the interconnection symbols given in Fig. 2-1 for interconnecting the problem from one machine to the next.

2-5. SCALE FACTORS

The variables of a problem to be solved on the analog computer can have any conceivable units. When placed on the computer, however,

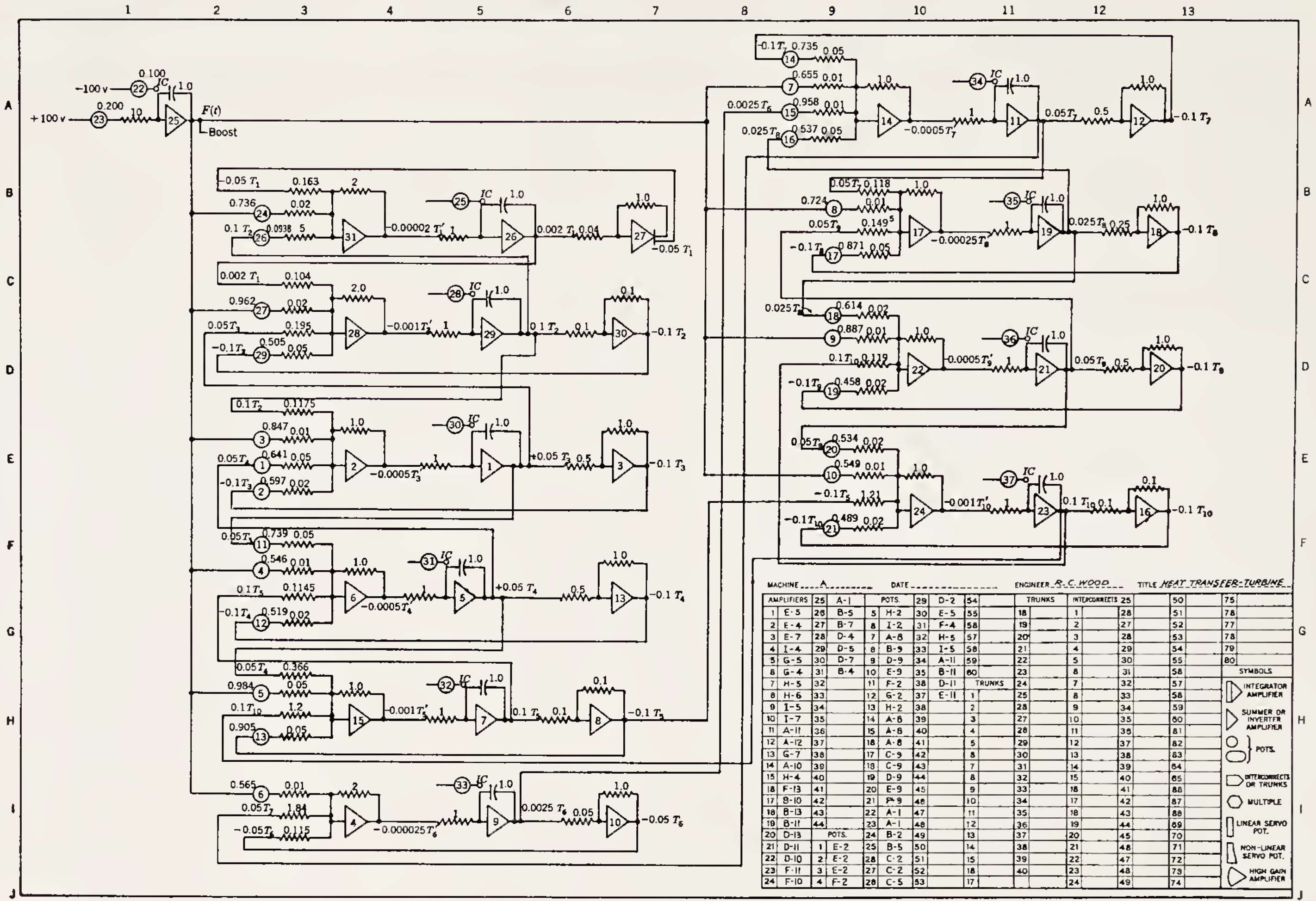


FIG. 2-5. Computer work sheet. (*Courtesy Ryan Aeronautical Co.*)

the variables must be expressed in volts. Hence, some scale factor is necessary to translate each variable from its natural unit to volts and back again. This scale factor is the number of volts representing one unit of the variable.

The most universally accepted computing range of analog computers is + and −100 volts. Therefore, all the problem variables must be scaled so as to be in this range. To do this, an estimation is made of the maximum value of the variables present in a problem. For example, a velocity x is estimated to have a maximum value of 400 ft/sec. Given an estimate, two methods can be followed to establish a scale factor. The first is to introduce an auxiliary set of variables. Thus, in our example of the velocity x, we can let $5V = x$; V then has a maximum value of 80 ft/sec. We can then let 1 volt equal 1 ft/sec for the variable V and still fall within our desired voltage range. This method seems somewhat artificial as it introduces an auxiliary set of variables which have no physical significance. The other method is to multiply or divide the problem variables by a suitable factor and compute $x/5$ in our example. This procedure retains the identity of the problem variables and the physical interpretation which goes with them. It should be emphasized that the scale factors refer only to the voltages coming out of the amplifiers and not to the input voltages. It is permissible to have scale factors on input voltages which exceed the 100-volt limit. It is required only that the sum of these voltages times their gains be less than 100 volts.

2-6. PROBLEM PREPARATION FOR THE ANALOG COMPUTER*

In the analysis of a given physical system the standard procedure calls for the application of the basic laws of physics to arrive at a set of differential equations. The solutions to these differential equations represent the motion of the physical variables as a function of time or some other independent variable. Representative problems include such things as the motion of a rigid body which possesses six degrees of freedom, three of translation, three of rotation; analysis of servo systems; etc. Having formulated the problem through application of physical principles, the analyst eventually arrives at a set of j differential equations in j unknowns, and of at most order n in any one of the j variables. Considering the case of a differential equation of order n in a single variable y, there results an equation of the form

$$f\left(\frac{d^ny}{dt^n},\frac{d^{n-1}y}{dt^{n-1}}, \ . \ . \ . \ ,\frac{dy}{dt},y,t\right) = 0 \tag{2-1}$$

---

* The presentation given in the introductory paragraphs of this section follows closely the excellent presentation given in pp. 165–167 of Ref. 7.

In the general case the functional relationships between the variable and its derivatives may be completely arbitrary. Thus, to accomplish a solution, it may be necessary to raise quantities to powers and perform multiplications, divisions, and any of the other elementary mathematical operations. A solution of Eq. (2-1) is accomplished when the variable y is expressed as a function of the independent variable t. In theory this solution may be accomplished by means of either n differentiations or n integrations. For the case of the n differentiations the process is illustrated by means of the following series of equations:

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}(y) \\ \frac{d^2y}{dt^2} &= \frac{d}{dt}\left(\frac{dy}{dt}\right) \\ \frac{d^ny}{dt^n} &= \frac{d}{dt}\left(\frac{d^{n-1}y}{dt^{n-1}}\right) \end{aligned} \tag{2-2}$$

If one considers the frequency-response characteristics of a differentiator, it is seen that the response is proportional to the forcing frequency. Thus, if one attempts a computation in the presence of noise (which is always the case), a computer interconnected by means of differentiators would be prone to be unstable. For this reason this method of programming is never used.

As an alternative to interconnecting the computer through differentiators, let us consider the solution of Eq. (2-1) by means of interconnecting n integrators. The solution by means of n successive integrators is illustrated by the following series of equations:

$$\begin{aligned} \frac{d^{n-1}y}{dt^{n-1}} &= \int \frac{d^ny}{dt^n}\,dt \\ \frac{d^{n-2}y}{dt^{n-2}} &= \int \frac{d^{n-1}y}{dt^{n-1}}\,dt \\ y &= \int \frac{dy}{dt}\,dt \end{aligned} \tag{2-3}$$

Thus the programming starts by first solving Eq. (2-1) for its highest derivative. All other terms are transferred to the right-hand side to obtain

$$\frac{d^ny}{dt^n} = f_2\left(\frac{d^{n-1}y}{dt^{n-1}},\frac{d^{n-2}y}{dt^{n-2}}, \cdots ,y,t\right) \tag{2-4}$$

Equation (2-4) is then integrated to obtain $d^{n-1}y/dt^{n-1}$, which in turn is integrated, and so forth, until after n integrations y is obtained.

To illustrate the procedure of interconnecting the computer by means of integrators, we consider the simple example of the damped harmonic oscillator.

The equation of motion of the damped harmonic oscillator is of the form

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t) \tag{2-5}$$

where m = mass
c = damping coefficient
k = spring rate
$f(t)$ = forcing function

To program this equation for the analog computer, we first solve for the highest derivative, to get

$$\frac{d^2x}{dt^2} = \frac{f(t)}{m} - \frac{c}{m}\frac{dx}{dt} - \frac{k}{m}x \tag{2-6}$$

Then dx/dt is obtained by integration, or

$$\frac{dx}{dt} = \int_0^t \left[\frac{f(t)}{m} - \frac{c}{m}\frac{dx}{dt} - \frac{k}{m}x\right] dt \tag{2-7}$$

and, integrating twice, we obtain

$$x = \int_0^t \frac{dx}{dt}\, dt \tag{2-8}$$

where both integrations are subject to the initial conditions

$$x = x_0 \qquad \text{and} \qquad \frac{dx}{dt} = \left(\frac{dx}{dt}\right)_0 \qquad \text{at } t = 0$$

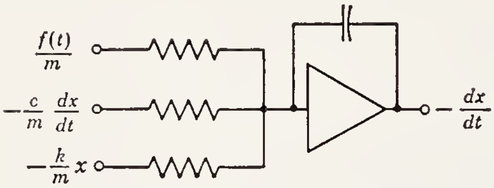


FIG. 2-6. Equipping the first integrator.

It is thus seen that after two integrations the variable x as a function of time has appeared. We wish now to draw a computer circuit diagram for this equation. To do this, we first examine the integrand of the integral from which dx/dt is obtained; this is Eq. (2-7). Thus, to obtain dx/dt, we must sum up the three quantities $\frac{f(t)}{m}$, $-\frac{c}{m}\frac{dx}{dt}$, and $-\frac{k}{m}x$, and integrate this sum. We do this as illustrated in Fig. 2-6, noting that there is a change in sign in passing through an amplifier. Next we produce x by a simple integration of $-dx/dt$, resulting in the circuit of Fig. 2-7. Now it will be

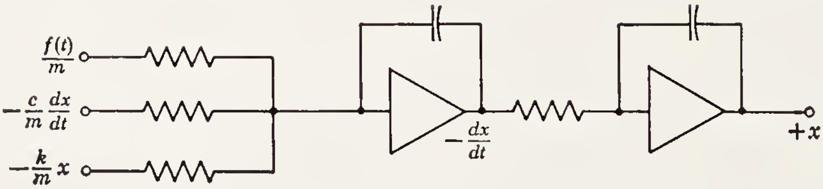


FIG. 2-7. Second step, obtaining $+x$ from $-dx/dt$.

noticed that $-\frac{c}{m}\frac{dx}{dt}$ and $-\frac{k}{m}x$ are required as inputs to the first integrator. The quantities $-dx/dt$ and $+x$ are available as the outputs of the first and second integrator, respectively. The quantity $-dx/dt$ is fed directly back to the grid of the first integrator with the input resistor adjusted to give a value c/m. The variable x must first be run through another amplifier, which acts as a sign inverter, and then to the grid of the first integrator, with its input resistor adjusted to give a value k/m. This results in the circuit of Fig. 2-8.

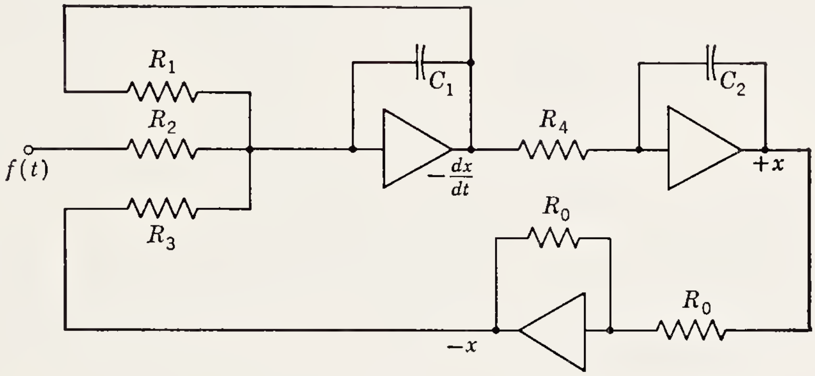


FIG. 2-8. Complete circuit for damped harmonic oscillator.

So far no mention has been made of the magnitude of the parameters involved in this equation. Only the suggestion has been made that the input resistors of the first integrator will be adjusted so as to give the correct magnitude. Suppose now that we take a specific case with parameters given as

$$\begin{aligned} m &= 5 \text{ lb sec}^2/\text{in.} \\ k &= 550 \text{ lb/in.} \\ c &= 10.0 \text{ lb sec/in.} \end{aligned}$$

and the initial conditions as

$$x = 0 \qquad \text{and} \qquad \frac{dx}{dt} = 0 \qquad \text{when } t = 0$$

The forcing function is specified as a step function applied at $t = 0$ and is of magnitude 600 lb. Thus

$$f(t) = \begin{cases} 0 & t < 0 \\ 600 & t \geq 0 \end{cases}$$

The constants are

$$\frac{f}{m} = \frac{600}{5} = 120 \text{ in./sec}^2 \qquad \frac{c}{m} = 2.0/\text{sec}$$

and

$$\frac{k}{m} = 110/\text{sec}^2$$

In numerical form the equation to be solved is

$$\frac{d^2x}{dt^2} = 120 - 2.0\frac{dx}{dt} - 110x \tag{2-9}$$

We now propose to select a set of scale factors such that the problem variables will remain within the bounds of the computer. It is estimated that the ranges of the variables will be approximately

$$0 \leq x \leq 2.0 \text{ in.} \qquad -20 < \frac{dx}{dt} < 20 \text{ in./sec}$$

$$-200 < \frac{d^2x}{dt^2} < 200 \text{ in./sec}$$

The scale factors selected are:

$$\text{Scale factor for } \frac{dx}{dt} \text{ is } \frac{100}{20} = 5.0$$

$$\text{Scale factor for } x \text{ is } \frac{100}{2.0} = 50$$

Now since d^2x/dt^2 does not appear in the circuit as the output of an amplifier, we need not select a scale factor for d^2x/dt^2. Using these scale factors, the scaled equation becomes

$$5.0\frac{d^2x}{dt^2} = 600 - 10\frac{dx}{dt} - 550x \tag{2-10}$$

Now to assign the values to the resistors: For this purpose we redraw the circuit and number the amplifiers 1, 2, 3, respectively, as shown in Fig. 2-9.

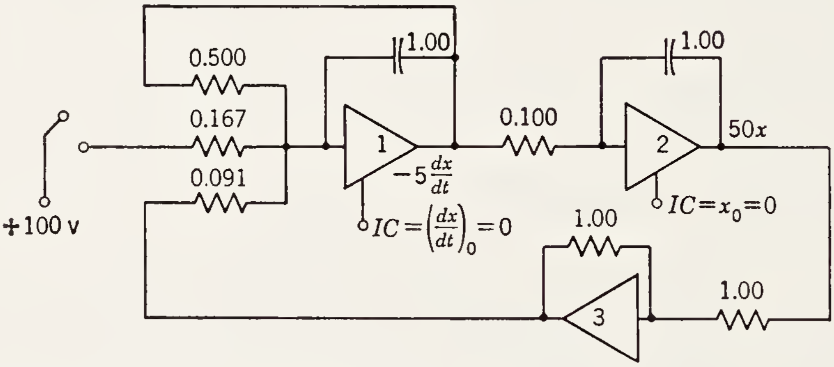


FIG. 2-9. Circuit for damped harmonic oscillator with fixed resistance values.

Starting with amplifier 1, we desire to have its output be 5.0 dx/dt. To accomplish this, we need to integrate the right-hand side of Eq. (2-10). The forcing function is determined as

$$100\frac{1}{R_2C_1s} = \frac{600}{s}$$

If $\qquad C_1 = 1\ \mu\text{f}$

then $\qquad R_2 = {}^{100}\!/_{600} = 0.167 \text{ megohm}$

It will be noticed that in this calculation if we let C be in microfarads and R in megohms we need not use the factors 10^{-6} and 10^{6}, respectively. Since the permissible scale factor on x is 50, we shall assume that $-50x$ is available as input to amplifier 1. This then determines the resistor R_3 as

$$(50x)\frac{1}{R_3C_1s} = \frac{550x}{s} \qquad \text{or} \qquad R_3 = \frac{50}{550} = 0.091 \text{ megohm}$$

Since the scale factor was adjusted so that the output of amplifier 1 would be 5.0 dx/dt, then the resistor R_1 is

$$5.0\frac{dx}{dt}\frac{1}{R_1C_1s} = \frac{10.0}{s}\frac{dx}{dt} \qquad \text{or} \qquad R_1 = \frac{5.0}{10.0} = 0.500 \text{ megohm}$$

This completes the programming of amplifier 1.

In order to obtain $50x$ out of amplifier 2, we let $5.0\frac{dx}{dt}\frac{1}{R_4C_2s} = 50x$, or $R_4 = 5.0/50$, assuming that $C_2 = 1\ \mu$f. Amplifier 3 serves only as a sign inverter; hence, the only requirements on the input and feedback resistors for this amplifier are that they be equal to each other. Since it is common practice to use 1-megohm resistors for feedback, we simply choose a 1-megohm input to match it.

The resistors R_1, R_2, and R_3 have values which would not in general be readily available on most computers. Therefore, in programming a problem for a computer on which the input resistors are not readily adjustable, it will be necessary to modify the circuit to include potentiometers. Figure 2-10 shows a diagram of the circuit for the damped harmonic oscillator, redrawn to include potentiometers.

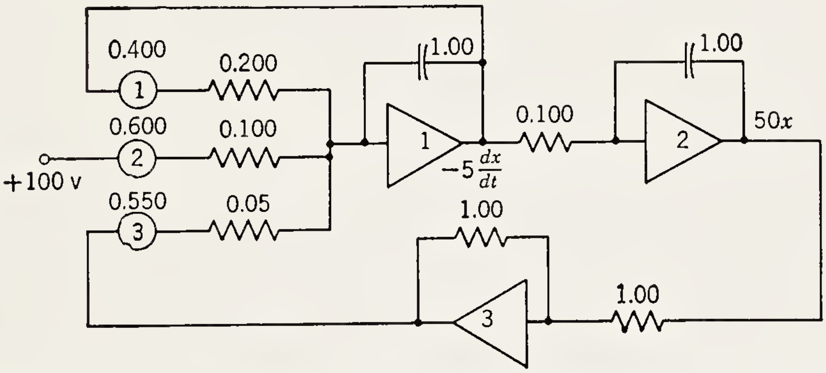


FIG. 2-10. Circuit diagram for damped harmonic oscillator using potentiometers and common values for resistors.

In the previous circuit we notice that the gain associated with resistance R_1 is $1/0.500 = 2.00$. Therefore, in Fig. 2-10 if we make $R_1 = 0.200$ megohm, giving a gain of 5, and set potentiometer 1 to the ratio $2.00/5.00 = 0.400$, then the required gain is established. Similarly:

1. The gain associated with R_2 is $1/0.167 = 6$; let $R_2 = 0.100$ megohm, giving a gain of 10; then set potentiometer 2 to $6/10 = 0.600$.
2. The gain associated with R_3 is $1/0.091 = 11$; let $R_3 = 0.050$ megohm, giving a gain of 20; then set potentiometer 3 to $11/20 = 0.550$.

These gains are obtainable on most computers. It should be noted that the gain of 20 associated with resistance R_3, however, would probably have to be obtained by paralleling two gains of 10. This concludes the discussion of the simple case of the harmonic oscillator.

Returning now to the general case, we isolate the highest derivative of Eq. (2-3) on the left-hand side of the equation. All other terms are moved to the right-hand side, as shown in Eq. (2-11).

$$\frac{d^ny}{dt^n} = f_1\left(\frac{d^{n-1}y}{dt^{n-1}},\frac{d^{n-2}y}{dt^{n-2}}, \cdots ,\frac{dy}{dt},y,t\right) \tag{2-11}$$

The highest derivative is thus expressed in terms of the lower derivatives and the variables themselves. The right-hand side of Eq. (2-11) may, therefore, be substituted as the integrand in the first of Eqs. (2-3). Since all the variables in this integrand have known initial values, an integrating amplifier may be assigned to perform the integration called for by the first of Eqs. (2-3) and the initial condition of this integrator set to an initial voltage required by the right-hand side of Eq. (2-11). As the independent variable voltage t starts to develop increasing values of time, the various derivatives and other variables and parameters of the problem will correspondingly change in conformance with Eq. (2-1). By connecting into the grid of the first integrator, through appropriate resistors, all the voltages which represent the variables on the right-hand side of Eq. (2-11), the output of that integrator will be at all times the $(n - 1)$st derivative. If the $(n - 1)$st derivative enters into Eq. (2-11) linearly, then the output voltage of the first integrator will be fed back into its own grid through an appropriate resistor to supply the $(n - 1)$st derivative called for by Eq. (2-11). Should the $(n - 1)$st derivative appear in a functional form other than linear, however, it will be necessary to form this required function before feeding it into the grid of the first integrator. So far only the first integrator has materialized. The suggestion has been made that the other voltages contributing to the input voltage on this integrator are present in the computer. The various derivatives, of course, must be provided by additional integrators, which step down the order of the derivative in successive steps. The biggest hurdle is overcome, in general, when the first integrator is properly equipped. After that, the output of this integrator is connected into the input grid of the second integrator, whose output, in turn, is the $(n - 2)$nd derivative. This is in accordance with the second of Eqs. (2-3). The initial setting to the second integrator will be the initial value of the

$(n - 2)$nd derivative. The output of the second integrator is operated on to produce the correct functional form required by Eq. (2-11). This output is then connected through an appropriate resistor into the grid of the first integrator to supply the function of the $(n - 2)$nd derivative called for in Eq. (2-11). The output of the second integrator is also connected into the grid of the third integrator to provide the $(n - 2)$nd derivative, etc. Each integrator also feeds the grid of the next integrator. Thus, at the nth integrator, the function y finally appears. This output is operated on to produce the functional form required by Eq. (2-11). It is then fed to the grid of the first integrator, and to any other point in the computer where the value y is required.

2-7. THE NEED FOR TIME TRANSFORMATION

Computations on the analog computer are made by virtue of connecting together a set of elements such as amplifiers, potentiometers, and servomultipliers, etc. When properly programmed, a formal analogy exists between the computer variables and the variables of the problem to be analyzed. Since the computer components in themselves represent dynamic elements which, above certain frequencies, depart from ideal characteristics, the analogy of the computer variables to the problem variables exists only when the problem frequencies are low compared with the computer-component break frequencies.

If this situation does not prevail, i.e., if the problem frequencies are such as to exceed the capabilities of the computer components, then a time transformation must be made which slows down the problem frequencies. Thus, if the highest problem frequency ω has a period of T sec, and if the problem is slowed down by a factor α, then the period corresponding to the highest problem frequency is changed to αT and the frequency is changed to ω/α. A number α can be found which makes the problem frequencies low compared with the dynamic characteristics of the computer, and, as a consequence, the analogy between the problem variables and the computer variables is valid. When the time scale of a problem is changed, the computer variables remain proportional to their corresponding variables in the physical system. The rate of problem solution is changed, however, so that the solution of the problem will be either slowed or accelerated.

Consider now the situation where the problem variables have extremely slow rates of change, such as the rate of cooling of a large body. Then the problem variables may require several minutes to several hours to reach a state of equilibrium. If no time transformation was made and the computer analogy established, the input voltages to the integrators would have to be extremely small. Inaccuracies in potentiometer settings, resistance values, multiplier null errors, and integrator drift would be of the same order of magnitude as the true voltages to be integrated, which would cause difficulty and mounting error. Thus, if the problem calls for integrating small voltages over a long period of time, the accuracy of the results is questionable. To avoid this situation, a time transformation which speeds up the computing time is required. In general a time transformation would be made to limit the time of solution of the problem to not more than 2 or 3 min of real time. The general rule is to make the computing time as short as possible, being careful not to make it so short that the dynamic characteristics of the computer are introduced.

2-8. MAKING THE TIME TRANSFORMATION

If t is the independent variable of the problem and τ is the computer time scale, then, since we desire to make a constant transformation, we have $\alpha t = \tau$. If α is greater than unity, the solution is slowed by the factor α. If α is less than unity, the solution is speeded up by the factor $1/\alpha$.

If the substitution $t = \tau/\alpha$ is made in the equations to be solved, the derivatives become

$$\frac{d}{dt} = \frac{d}{d(\tau/\alpha)} = \alpha \frac{d}{d\tau} \tag{2-12}$$

and

$$\frac{d^2}{dt^2} = \frac{d}{dt}\left(\alpha \frac{d}{d\tau}\right) = \frac{d}{d(\tau/\alpha)}\left(\alpha \frac{d}{d\tau}\right) = \alpha^2 \frac{d^2}{d\tau^2}$$

Hence in general

$$\frac{d^n}{dt^n} = \alpha^n \frac{d^n}{d\tau^n} \tag{2-13}$$

This may be summarized as a convenient rule of thumb: To slow down time by a factor of α, multiply the nth derivative by α^n. To speed up time by a factor of α, divide the nth derivative by α^n.

To illustrate the mechanics involved in making the time transformation, let us again consider the equation of motion of the damped harmonic oscillator,

$$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = f(t) \tag{2-5}$$

Suppose we consider the case where

$$m = 0.25 \text{ slug} \qquad \text{and} \qquad f(t) = \begin{cases} 0 & t < 0 \\ 10{,}000 & t \geq 0 \end{cases}$$

$$c = 10.0 \text{ lb sec/in.}$$
$$k = 2{,}000 \text{ lb/in.}$$

The constants are

$$\frac{f}{m} = \frac{10{,}000}{0.25} = 40{,}000 \text{ in./sec}^2$$

$$\frac{c}{m} = 40/\text{sec}$$

and

$$\frac{k}{m} = 8{,}000/\text{sec}^2$$

In numerical form the equation to be solved is

$$\frac{d^2x}{dt^2} = 40{,}000 - 40\frac{dx}{dt} - 8{,}000x \tag{2-14}$$

The ranges of the variables are estimated to be

$$0 \leq x < 10 \text{ in.} \qquad -500 < \frac{dx}{dt} < 500 \text{ in./sec}$$

Thus the scale factors selected are:

For $\frac{dx}{dt}$: $^{100}\!/_{500} = 0.20$

For x: $^{100}\!/_{10} = 10$

and the scaled equation becomes

$$\frac{1}{5}\frac{d^2x}{dt^2} = 8{,}000 - 8\frac{dx}{dt} - 1{,}600x \tag{2-15}$$

Following the procedure outlined in Sec. 2-6, the circuit of Fig. 2-11 is obtained. It will be noticed that the gains associated with amplifier 1 vary from 40 to 160 and are therefore higher than desired.

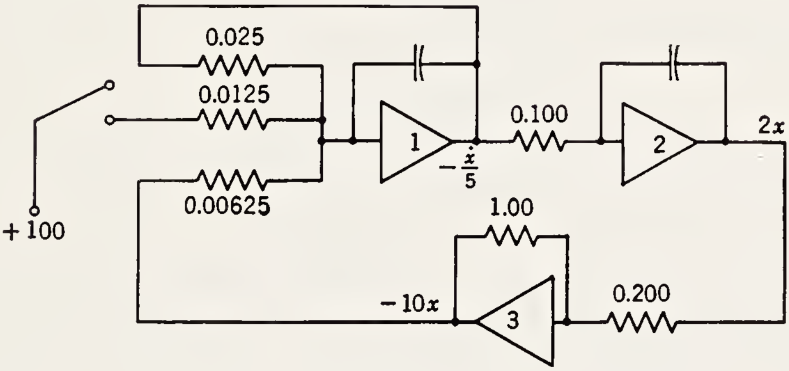


FIG. 2-11. Circuit for solving Eq. (2-15) with no time transformation.

To relieve this situation, suppose that we make a time transformation to slow the time down by a factor of 10. To do this, we multiply the second derivative by 100 and the first derivative by 10, to obtain

$$20\frac{d^2x}{d\tau^2} = 8{,}000 - 80\frac{dx}{d\tau^2} - 1{,}600x \qquad (2\text{-}16)$$

Using the same scale factors, we obtain the scaled equation as

$$\frac{1}{5}\frac{d^2x}{d\tau^2} = 80 - 0.80\frac{dx}{d\tau} - 16x \qquad (2\text{-}17)$$

Figure 2-12 shows the circuit for this equation. Note that the magnitude of the gains associated with amplifier 1 has been decreased so that the maximum gain is 8.

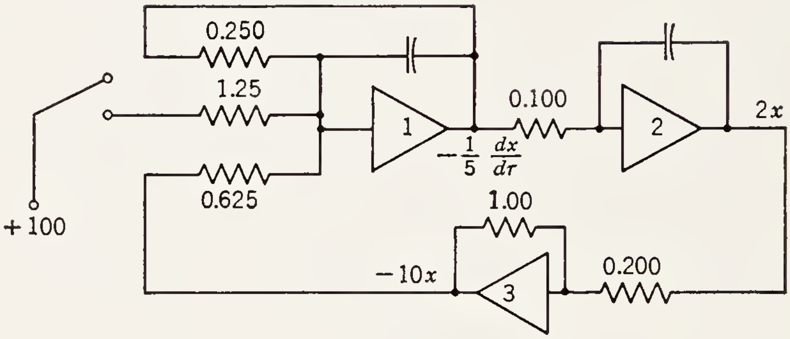


FIG. 2-12. Circuit for solving Eq. (2-15) at one-tenth real time.

The time-scale transformation can be looked at from a slightly different point of view. Suppose that we introduce dimensionless time in the differential equation to be solved. For example, consider Eq. (2-5),

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t) \qquad (2\text{-}5)$$

Divide through by m, to obtain

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{f(t)}{m} \qquad (2\text{-}18)$$

Now the undamped natural frequency is defined as

$$\omega_n = \sqrt{\frac{k}{m}}$$

and the damping ratio is defined as

$$\xi = \frac{c}{2\sqrt{mk}}$$

and so the equation may be written as

$$\frac{d^2x}{dt^2} + 2\xi\omega_n\frac{dx}{dt} + \omega_n{}^2x = \frac{f(t)}{m} \qquad (2\text{-}19)$$

If we now define nondimensional time τ as $\omega_n t = \tau$, then

$$\frac{d}{dt} = \omega_n \frac{d}{d\tau} \tag{2-20}$$

and

$$\frac{d^2}{dt^2} = \omega_n{}^2 \frac{d^2}{d\tau^2} \tag{2-21}$$

Thus Eq. (2-19) becomes

$$\omega_n{}^2 \frac{d^2x}{d\tau^2} + 2\xi\omega_n{}^2 \frac{dx}{d\tau} + \omega_n{}^2 x = \frac{f(t)}{m} \tag{2-22}$$

or

$$\frac{d^2x}{d\tau^2} + 2\xi \frac{dx}{d\tau} + x = \frac{f(\tau/\omega_n)}{k} \tag{2-23}$$

This equation can be programmed to have a very simple circuit, the solution of which in nondimensional time contains the solutions for all time scales. This is true since the solutions for all other time scales can be obtained by dilating the time scale of the nondimensional solution by the factor $1/\omega_n$. It should be noticed that if the forcing function is time variant then it will be necessary to expand or contract its time scale accordingly.

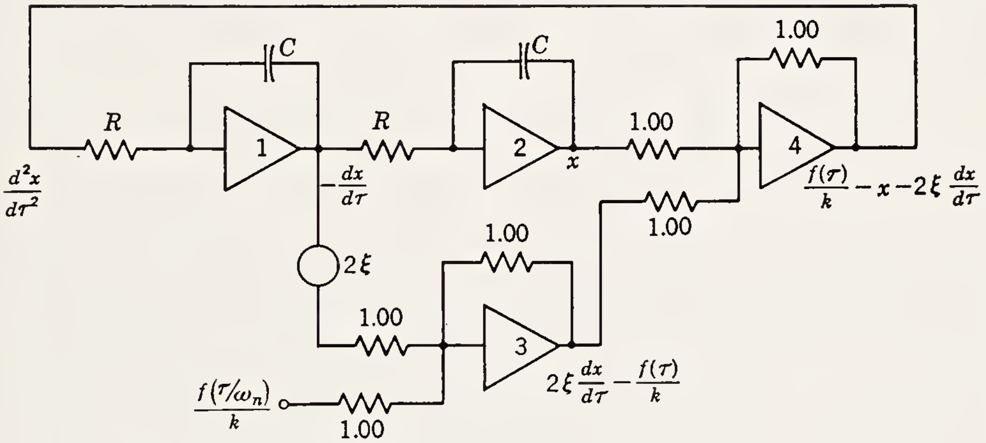

1.00
C
C
R
1
R
2
1.00
4
$\frac{d^2x}{d\tau^2}$
$-\frac{dx}{d\tau}$
x
$\frac{f(\tau)}{k} - x - 2\xi \frac{dx}{d\tau}$
1.00
1.00
2ξ
1.00
3
$2\xi \frac{dx}{d\tau} - \frac{f(\tau)}{k}$
$\frac{f(\tau/\omega_n)}{k}$
1.00

FIG. 2-13. Circuit for solution of Eq. (2-23) in nondimensional time.

A circuit for the solution of Eq. (2-23) in nondimensional time is given in Fig. 2-13. In this circuit the time constant for each integrator is RC sec. The choice of RC will determine the number of seconds of computer time equivalent to one unit of dimensionless time τ. If $R = 1$ megohm and $C = 1\ \mu$f, then 1 sec of computer time equals one unit of dimensionless time τ. This method of visualizing the time transformation has the advantage that in any problem so programmed a time transformation may be achieved by simply changing the RC product of the integrators. The technique of nondimensionalizing with respect to time is applicable whenever the differential equation is homogeneous under a dilation of time scale.*

2-9. PROGRAMMING TRANSFER FUNCTIONS

Frequently problems arise where the information is given in the form of a transfer function

$$\frac{E_o}{E_i} = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0} \tag{2-24}$$

The methods previously presented for programming are not directly applicable to this problem, since there seemingly is no way of isolating the highest derivative. Suppose that we cross-multiply both sides of this equation and set the resulting two equations equal to each other and to an auxiliary variable M. This process results in

$$\begin{aligned} \frac{E_i}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0} &= M \\ &= \frac{E_o}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \end{aligned} \tag{2-25}$$

Now, interpreting the operator s as indicating the operation of differentiation, there is obtained

$$b_n \frac{d^n}{dt^n} M + b_{n-1}\frac{d^{n-1}}{dt^{n-1}} M + \cdots + b_1 \frac{d}{dt} M + b_0 M = e_i \tag{2-26}$$

and

$$a_n \frac{d^n}{dt^n} M + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} M + \cdots + a_1 \frac{d}{dt} M + a_0 M = e_o \tag{2-27}$$

On examining Eqs. (2-26) and (2-27), it is noticed that Eq. (2-27) for the output voltage e_o consists of terms which are proportional to M and derivatives of M up to order n. If now Eq. (2-26) is solved for the highest derivative in M and programmed, there will be obtained voltages proportional to M and all the derivatives of M up to order n. Thus all the information required to generate e_o is contained in the program of Eq. (2-26). Solving Eq. (2-26) for the highest derivative, we obtain

$$\frac{d^n}{dt^n} M = \frac{e_i}{b_n} - \frac{b_{n-1}}{b_n}\frac{d^{n-1}}{dt^{n-1}} M - \cdots - \frac{b_1}{b_n}\frac{d}{dt} M - \frac{b_0}{b_n} M \tag{2-28}$$

which has a very simple circuit diagram. In particular, if Eq. (2-28) is programmed so that voltages proportional to all n derivatives are present, then the desired output e_o can be obtained by a simple summing process. A typical circuit for a high-order transfer function is illustrated as Fig.

---

* By definition a function is homogeneous of degree n under dilation if $f(at) = a^n f(t)$.

2-14. In this circuit, if n is the highest power of s appearing in the transfer function, then n integrators will be required, each stepping down the derivative by 1. In addition, two summing amplifiers will be required to form the sum of the terms in $(d^n/dt^n)M$. If we now examine Eq. (2-27) for e_o, we notice that all the information required to form e_o is present. To form e_o, it is necessary to take into account the arbitrariness of the sign of the coefficients a_n. This can be accomplished by using two summing amplifiers.

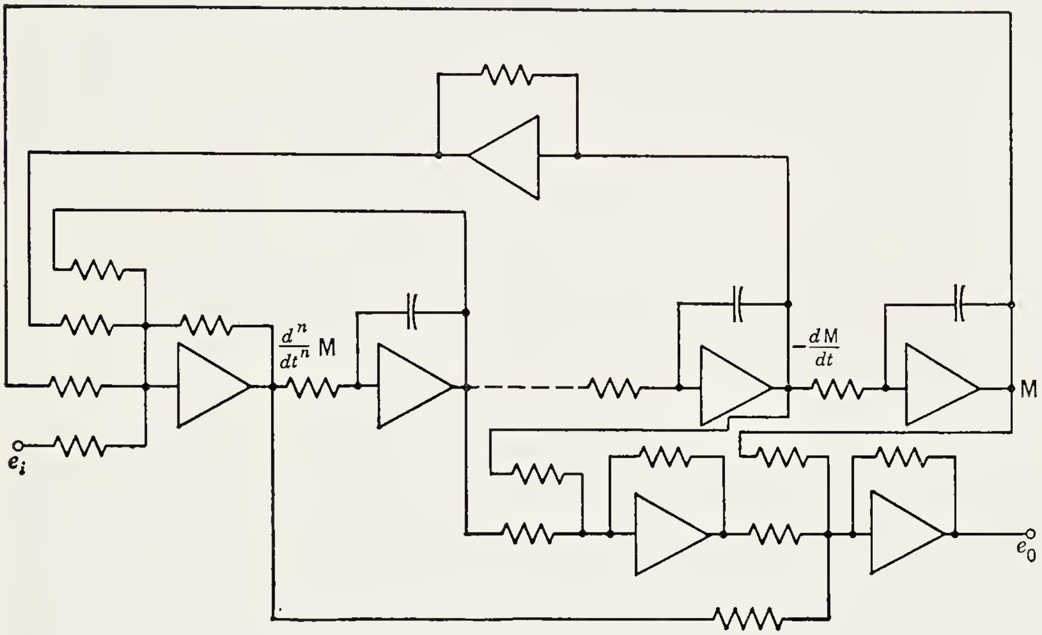


FIG. 2-14. Typical circuit for simulation of a transfer function e_o/e_i.

If the highest power of s in the numerator is less than the highest power of s in the denominator, then one of the summing amplifiers in the sum $(d^n/dt^n)M$ can be dispensed with, as in this case the integrator can be used to do the summing. Depending on the computer used, it may be desirable to include potentiometers at various points in this circuit.

2-10. SIMULATION OF A HIGH-ORDER TRANSFER FUNCTION WHICH REQUIRES TIME TRANSFORMATION

The methods given in the last two sections will be illustrated by the following example: It is required to devise a circuit for realizing the transfer function

$$\frac{E_o}{E_i} = \frac{1.06s}{1 + 1.02s + 2.31 \times 10^{-2}s^2 + 2.45 \times 10^{-4}s^3 + 1.26 \times 10^{-6}s^4 + 2.67 \times 10^{-9}s^5} \tag{2-29}$$

Following the procedure given in Sec. 2-9, we cross-multiply to get

$$\frac{E_o}{1.06s} = M = \frac{E_i}{1 + 1.02s + 2.31 \times 10^{-2}s^2 + 2.45 \times 10^{-4}s^3 + 1.26 \times 10^{-6}s^4 + 2.67 \times 10^{-9}s^5} \tag{2-30}$$

Converting back to the differential equation by interpreting the operator s as indicating the operation of differentiation, there is obtained*

$$2.67 \times 10^{-9}\overset{5}{M} + 1.26 \times 10^{-6}\overset{4}{M} + 2.45 \times 10^{-4}\overset{3}{M} + 2.31 \times 10^{-2}\overset{2}{M} + 1.02\overset{1}{M} + M = e_i$$

and

$$e_o = 1.06\overset{1}{M}$$

Solving the first of these for the highest derivative, we obtain

$$\overset{5}{M} = 3.75 \times 10^8 e_1 - 4.70 \times 10^2\overset{4}{M} - 9.16 \times 10^5\overset{3}{M} - 8.66 \times 10^6\overset{2}{M} - 3.82 \times 10^8\overset{1}{M} - 3.75 \times 10^8 M \tag{2-31}$$

Inspecting this equation, it is seen that, going from the fifth derivative to the fourth derivative, we need a gain of 470, from the fourth to the third a gain of 1,950, from the third to the second 9.45, from the second to the first 44.1, and from the first to the variable a gain of 0.983. The gains of 470 and 1,950 in the first two stages would be difficult to achieve and still have the remaining circuit stable. Therefore a time transformation is indicated.

Suppose that we run the problem at a time scale of $\frac{1}{100}$; that is, it takes 100 sec of computer time to equal 1 sec of real time. To make the time transformation to slow time, multiply the n derivative by $(100)^n$, to obtain

$$10^{10}\overset{5}{M} = 3.75 \times 10^8 e_i - 4.70 \times 10^{10}\overset{4}{M} - 9.16 \times 10^{11}\overset{3}{M} - 8.66 \times 10^{10}\overset{2}{M} - 3.82 \times 10^{10}\overset{1}{M} - 3.75 \times 10^8 M$$

Dividing by 10^{10}, we have

$$\overset{5}{M} = 0.0375e_i - 4.70\overset{4}{M} - 91.6\overset{3}{M} - 8.66\overset{2}{M} - 3.82\overset{1}{M} - 0.0375M \tag{2-32}$$

---

* Where

$$\overset{n}{M} \equiv \frac{d^n}{dt^n} M$$

Inspection of this equation shows that the greatest gain per stage has been reduced to 19.5, which is a practical value. The diagram of the problem is shown in Fig. 2-15. In this case, since $e_o = \frac{1}{1.06M}$ and the output of amplifier 9 is $\frac{1}{3.88M}$, then the output of amplifier 9 is 3.88/1.06 = 3.66 times the desired output. Calculation of the resistance values to obtain the desired gains in this circuit is left as an exercise for the reader.

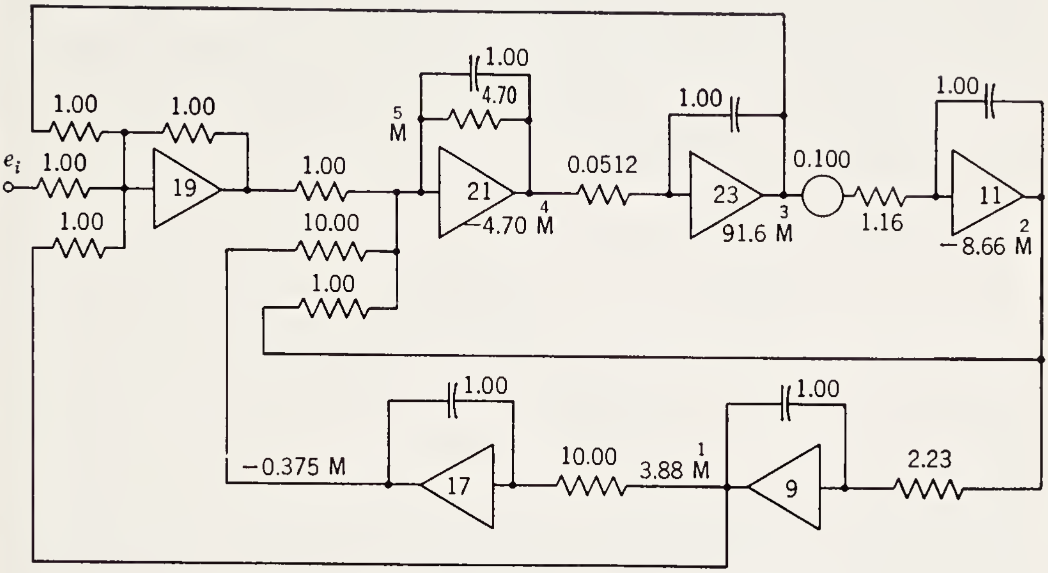


FIG. 2-15. Circuit for simulating the transfer function

$$\frac{E_o}{E_i} = \frac{1.06s}{1 + 1.02s + 2.31 \times 10^{-2}s^2 + 2.45 \times 10^{-4}s^3 + 1.26 \times 10^{-6}s^4 + 2.67 \times 10^{-9}s^5}$$

2-11. PROGRAMMING A PROBLEM WHICH REQUIRES MULTIPLICATION

The problems programmed up to this point have involved only linear operations. To illustrate the use of servomultiplier equipment, a problem which requires multiplication will be programmed. Since this is the reader's first encounter with programming problems involving multiplication, a detailed step-by-step procedure will be given.

The problem selected for this example is the well-known equation of Van der Pol,

$$\frac{d^2y}{dt^2} + \mu(y^2 - 1)\frac{dy}{dt} + y = 0 \qquad (2\text{-}33)$$

In this equation μ is a constant parameter which may be used, for instance, to describe the build-up of oscillations in certain nonlinear electrical circuits. The Van der Pol equation is seen to resemble the performance equation of a damped harmonic oscillator but has a nonlinear damping term which will tend to build up the amplitude of the oscillations for small values of y and to decrease the amplitude for larger values of y. If an oscillator of this type is given a small initial displacement y, periodic oscillations of constant amplitude will result after an initial transient. To program this equation, we first start by solving for the second derivative, to obtain

$$\frac{d^2y}{dt^2} = -\mu(y^2 - 1)\frac{dy}{dt} - y \tag{2-34}$$

$$\frac{dy}{dt} = \int_0^t \left(-\mu y^2 \frac{dy}{dt} + \mu \frac{dy}{dt} - y\right) dt \tag{2-35}$$

and

$$y = \int_0^t \left(\frac{dy}{dt}\right) dt \tag{2-36}$$

with suitable initial conditions. To select the scale factors, we note that the parameter μ is unspecified. It is therefore possible to consider several cases; these are $0 < \mu \leq 1$, $1 < \mu \leq 10$, etc. In setting up the problem we should consider programming the circuit so that changing μ by a factor of 10 can be readily accomplished. To select the scale factors, it is given that for $\mu = 0.1$ the magnitude of y is approximately 2. Thus, if we select a scale factor of 10, we should obtain approximately 20 volts maximum. For $\mu < 0.1$ the maximum is greater, and for $\mu > 0.1$ the maximum is less. In the range of $0 < \mu \leq 1$ a scale factor of $10y$ would thus appear to be reasonable. Since nothing is known about dy/dt, we shall calculate it directly. It is worth noting at this point that frequently in many problems estimates of magnitudes and time responses cannot readily be made. To get round this situation, one can proceed with the information at hand, and as the programming progresses, if the scale factors and time responses are incorrect, this will usually (but not always) manifest itself in the form of unreasonably large gains. The final test of magnitudes can be accomplished when the problem is first wired and during the first computer runs. The magnitudes of the voltages out of the various amplifiers can be checked, and scaled up or down accordingly, until reasonable values are obtained. The time responses can be checked after the scale factors have been satisfactorily selected by making a time transformation and cutting the time scale by a factor of 2 and comparing the results. If the results obtained in this process agree with the results obtained without the time transformation, it is assumed that the original time scale is correct.

Examining Eq. (2-35), it is seen that to obtain dy/dt we must sum up the quantities $-\mu y^2(dy/dt)$, $\mu(dy/dt)$, and $-y$ and integrate this sum. The first of these quantities involves the product of the square of the function times the first derivative. For the present, to equip the first integrator, we assume that the quantities $-y^2 \frac{dy}{dt}$, $\frac{dy}{dt}$, and $-y$ are available. Thus Fig. 2-16 for the first integrator is obtained. Since we have agreed to let the output of the first integrator be $-dy/dt$, we can select the input resistors all to be 1 megohm if the feedback capacitor is 1 μf.

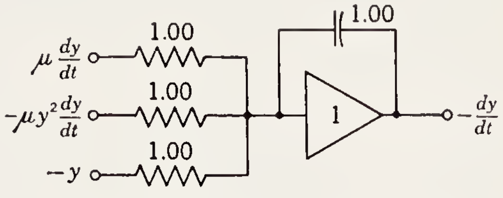


Fig. 2-16. Equipping the first integrator for the solution of Van der Pol's equation.

To obtain the voltages required as inputs to integrator 1, we notice that the quantity $-y$ is called for. This may be obtained by another integration followed by a sign inversion. Also $+\mu(dy/dt)$ is called for, which can be obtained by a sign inversion of $-dy/dt$. Thus the circuit of Fig. 2-17 is obtained. Note that potentiometers 1 and 3 have been added in this circuit. Since $0 < \mu \leq 1$, potentiometer 1 can be set to read directly the value μ. Integrator 2 has been given a gain of 10 to make its output correspond to the desired $10y$. Potentiometer 3 steps down $10y$ to y, and amplifiers 3 and 11 have unity gain but provide the desired sign inversion.

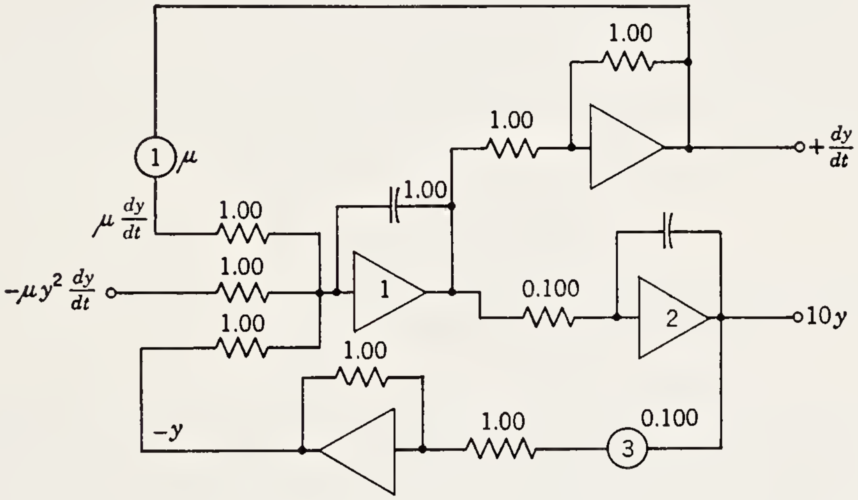


Fig. 2-17. Generation of $\mu(dy/dt)$ and $-y$ for solution of Van der Pol's equation.

We are now left only with the task of forming the function $y^2(dy/dt)$. To do this using the minimum amount of equipment, we multiply dy/dt by y to obtain $y(dy/dt)$ and then multiply this product by y to obtain $y^2(dy/dt)$. This can be accomplished on one servomultiplier if we drive the motor of the servomultiplier by $10y$ and put $\pm dy/dt$ across the ends of the first multiplier potentiometer, to obtain the product $\frac{y}{10}\frac{dy}{dt}$. If now ± 10 times this product, or $\pm y(dy/dt)$, is placed across the second multiplying potentiometer, we shall multiply by $10y$ again to obtain $\frac{y^2}{10}\frac{dy}{dt}$. Inverting this quantity and adding potentiometer 2, which is set to the value μ, we have the final circuit shown in Fig. 2-18. It should be noted that since both multiplying potentiometers are loaded by 100-kilohm input resistors to an amplifier, the servomultiplier follow-up has to be loaded similarly. This circuit can also be made to give solutions for $1 < \mu \leq 10$ by changing the input resistors of integrator 1 fed by potentiometers 1 and 2 to 100 kilohms to give a gain of 10.

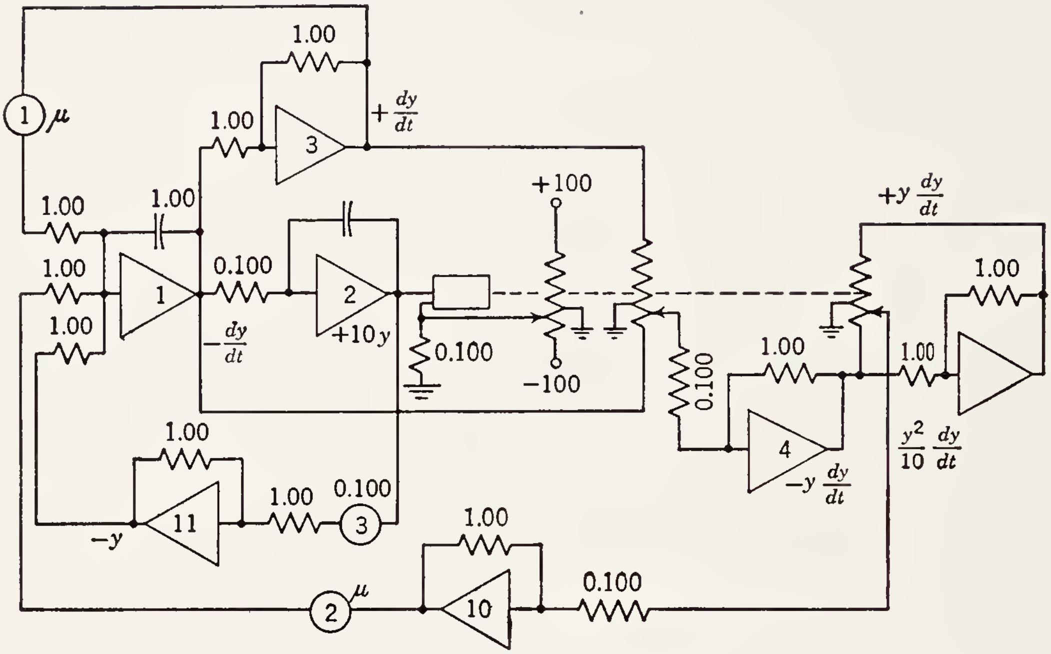


FIG. 2-18. Circuit for solution of Van der Pol's equation

$$\frac{d^2y}{dt^2} = -\mu y^2 \frac{dy}{dt} + \mu \frac{dy}{dt} - y$$

(*Note:* The case shown is for $\mu < 1$; for $1 < \mu \leq 10$ the resistor fed by potentiometers 1 and 2 into amplifier 1 should be changed to 0.100.)

2-12. GENERALIZED INTEGRATION

Since normally the analog computer performs integration only with respect to real time, it is necessary to resort to special techniques to integrate with respect to a dependent variable. The usual technique involves using or generating the time derivative of the variable in question, in order to satisfy the relationship

$$y = \int f(x)\,dx = \int f(x)\frac{dx}{dt}\,dt \tag{2-37}$$

A simple circuit for performing this integration is shown in Fig. 2-19.

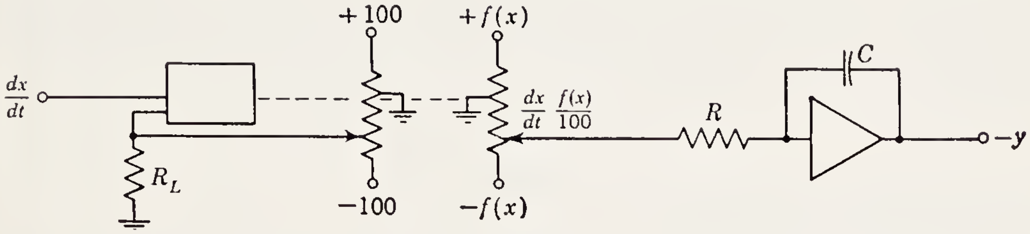


FIG. 2-19. Circuit for integrating with respect to a dependent variable.

In this circuit a servomultiplier is used to form the product $(dx/dt)f(x)$ and since ±100 volts is used across the follow-up potentiometer a factor of $\frac{1}{100}$ is introduced in the multiplication. This product is integrated to form $-y$. By use of this technique, which is most easily applied when the derivative of the quantity concerned appears in the problem, one can generate a number of standard functions. For example, consider the function e^x. Here we have $(d/dt)e^x = e^x(dx/dt)$; thus, if the time derivative of a function x is available in the computer and we desire to form the function e^x, the foregoing technique may be applied. Figure 2-20 shows the resulting circuit.

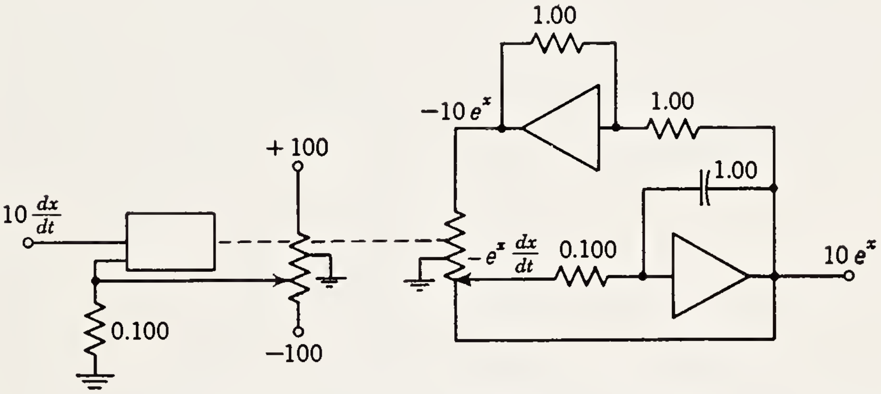


FIG. 2-20. Circuit for generating the function e^x.

The preceding example illustrates a general procedure which may be applied for the generation of analytic functions. The technique depends on the fact that the simple analytic functions are themselves solutions of differential equations. Thus, if it is desired to form, say, $y = \ln x$, we first form the differential equation whose solution is $y = \ln x$. To obtain the differential equation, we differentiate to obtain

$$\frac{dy}{dt} = \frac{1}{x}\frac{dx}{dt} \tag{2-38}$$

Thus

$$y = \int_0^t \frac{1}{x}\frac{dx}{dt}\,dt$$

Now, if dx/dt is available in the computer, a simple circuit can be used to form $y = \ln x$. Since, however, this involves division, which is as yet unexplained, the programming of this circuit is left as an exercise (see Prob. 6-1).

2-13. PROGRAMMING A SET OF SIMULTANEOUS DIFFERENTIAL EQUATIONS

In the study of dynamical systems with more than one degree of freedom there results a system of simultaneous differential equations. It is the purpose of this section to illustrate the application of the preceding method to the programming of a problem which involves a system of simultaneous differential equations.

Since in solving a problem on the analog computer it is customary that the analyst be given a statement of the problem so that he may have a physical "feel" for the system, we shall give a rather complete statement of this problem. The problem selected is to find the influence of gun recoil on the motion of an airplane.*

Statement of Problem. A large-caliber gun is mounted on the fuselage of an airplane at a distance of 7 ft behind the center of gravity. It fires 40 rounds per minute vertically upward, the recoil per shot being 10^3 lb, lasting for 0.3 sec, and causing an average force of 200 lb acting vertically downward. Given the data below, the problem is to find $\dot{\theta} = q$, or the angular pitching velocity about the Y axis due to gunfire, on the assumption that no corrective influence is exerted by the pilot.

Technical Data

Gross weight = 4,700 lb
Mass = 146.2 slugs
Span = 38 ft
Length = 27 ft 7 in.
Max speed = 172 mph

$$X_u = \frac{\partial X}{\partial u} = -0.0739$$

$$X_w = \frac{\partial X}{\partial w} = 0.0935$$

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* N. W. McLachlan, "Complex Variable and Transform Calculus," p. 203, Cambridge University Press, New York, 1953.

$$X_q = \frac{\partial X}{\partial q} = 0$$

$$Z_u = \frac{\partial Z}{\partial u} = -0.323$$

$$Z_w = \frac{\partial Z}{\partial w} = -2.28$$

$$Z_q = \frac{\partial Z}{\partial q} = 0$$

$$Z_G = \text{vertical force due to gun} = \frac{200}{146.2} = 1.37$$

$$M_u = \frac{\partial M}{\partial u} = 0$$

$$M_w = \frac{\partial M}{\partial w} = -1.4$$

$$M_q = \frac{\partial M}{\partial q} = -160$$

$$M_G = \text{moment due to gunfire} = \frac{200 \times 7}{146.2} = \frac{1{,}400}{146.2} = 9.56$$

$$K_B{}^2 = 38.6$$

$$U = \text{forward velocity} = 252 \text{ ft/sec}$$

where X, Z = forces in directions of x, z axes, respectively
U, W = velocities in directions of axes
u, w = incremental velocities due to disturbance
M_u, M_w, M_q = moments about Y axis due to velocities u, w and to pitch rate q, respectively
K_B = radius of gyration about Y axis

In accordance with aeronautical practice the forces and moments are given per unit mass of the airplane, the unit of mass being 1 slug.

Equations of Motion. The equations of motion along the x axis and the z axis and about the y axis (pitching) are, respectively,

$$\begin{aligned}
&\frac{du}{dt} - u\frac{\partial X}{\partial u} - w\frac{\partial X}{\partial w} + g\theta = 0 \\
&\frac{dw}{dt} - u\frac{\partial Z}{\partial u} - w\frac{\partial Z}{\partial w} - U\frac{d\theta}{dt} = Z_G \qquad \text{force due to gun recoil} \qquad (2\text{-}39) \\
&-w\frac{\partial M}{\partial w} + K_B{}^2\frac{d^2\theta}{dt^2} - \frac{M}{q}\frac{d\theta}{dt} = M_G \qquad \text{moment due to gun recoil}
\end{aligned}$$

Substituting the numerical values into these equations and using the dot notation, we obtain, on solving for the highest derivatives,

$$\begin{aligned}
\dot{u} &= -0.0739u + 0.0935w - 32.2\theta \\
\dot{w} &= 1.37 - 2.28w - 0.323u + 252\dot{\theta} \qquad (2\text{-}40) \\
38.6\ddot{\theta} &= 9.56 - 160\dot{\theta} - 1.4w
\end{aligned}$$

For the purpose of selecting scale factors, the magnitudes of the variables in the problem are estimated to have maximum values as follows:

$$\begin{array}{ll} u < 300 \text{ ft/sec} & w < 50 \text{ ft/sec} \\ \theta < 0.15 \text{ radian} & \dot{\theta} < 0.10 \text{ radian/sec} \\ \dot{u} < 10 \text{ ft/sec}^2 & \ddot{\theta} < 1 \text{ radian/sec}^2 \\ \dot{w} < 10 \text{ ft/sec}^2 & \end{array}$$

Thus, to stay within the ±100-volt operating range, we select the scale factors as

$$\frac{U}{3},\ 500\theta,\ 2w,\ 500\dot{\theta},\ 10\dot{u},\ 10\dot{w},\ 50\ddot{\theta}$$

Rewriting Eq. (2-40) in terms of the scale factors, we obtain

$$\begin{aligned} 10\dot{u} &= -0.739u + 0.935w - 322\theta \\ 10\dot{w} &= 13.7 - 22.8w - 3.23u + 2{,}520\dot{\theta} \\ 50\ddot{\theta} &= 12.4 - 207\dot{\theta} - 1.81w \end{aligned} \tag{2-41}$$

To program these equations, it is noted that the first can be thought of as the equation defining u, while the second defines w, and the third θ. In the general case it is not necessarily obvious which variable a particular equation, of a set of simultaneous equations, defines. In the set of Eqs. (2-41) the choice is fairly obvious, since if we had selected the first equation to define w, then to obtain $\dot{w}$ for the second equation would have required a differentiation. Similarly, if the second equation had been used to define θ, a differentiation would have been required to obtain θ for the third equation. Hence the general rule is to select, from the set of equations, an equation which contains at least the highest existing derivative of a given variable as the equation from which that variable is obtained by repeated integration. It may happen that two or more equations of the set each contain the highest derivative of the two or more variables. In this case consideration should be given to the simplest set to solve.

Having selected the equations which are to define the variables of the problem, the equations are individually solved so as to isolate the highest derivative of the defined variable on the left-hand side. The programming procedure is then carried out in exactly the same manner as that used for a single equation. That is, for each equation, the terms making up the right-hand side of the equation are assumed to be available in the computer. With this assumption it is possible to equip the first integrator for each variable. Thus the output of this first integrator will be the first integral of the highest derivative. The programming process then continues through more integrations until the variable itself is produced. In the case of physical systems, this usually involves at most two integrations per variable. Having thus produced all the variables and their derivatives to all orders required, the programming is completed by interconnecting the degrees of freedom through the appropriate functional relations called for in the equations to be solved.

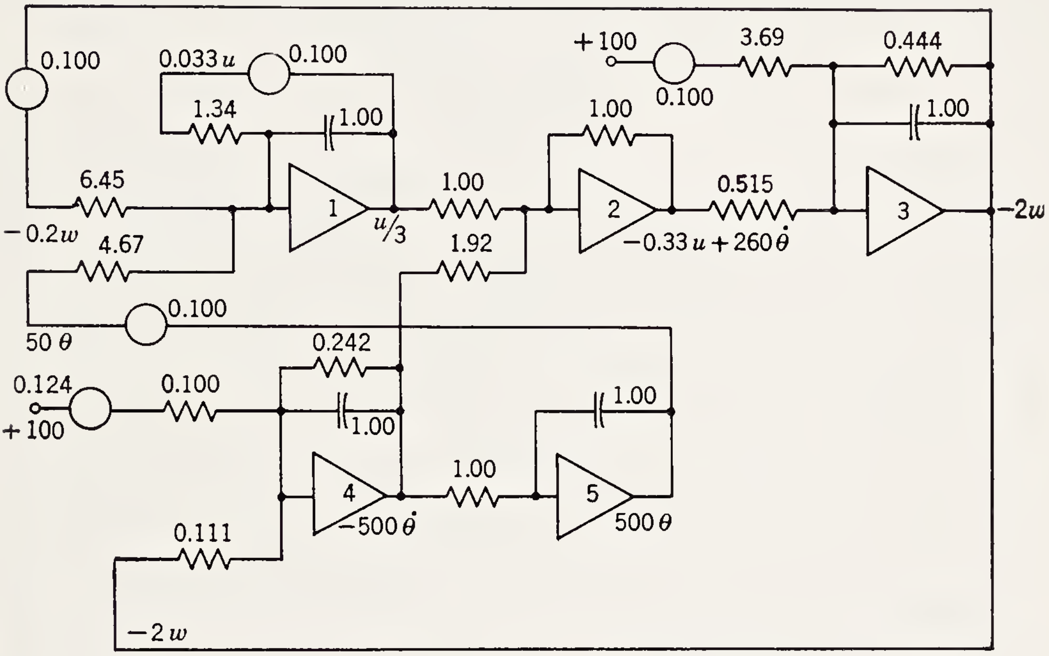

FIG. 2-21. Circuit diagram for solution of set of equations describing the influence of gun recoil on the pitching motion of an airplane.

Figure 2-21 illustrates this process for the example considered in this section. The derivation of the circuit details is left as an exercise for the reader.

2-14. PROGRAMMING A PROBLEM INVOLVING TRIGONOMETRIC OPERATIONS

As an example of a system which involves a trigonometric function, let us consider the equation of motion of a simple pendulum of variable length.*

$$\ddot{\theta} + \frac{2}{l}\frac{dl}{dt}\dot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{2-42}$$

where θ = angular displacement from equilibrium
l = variable length of pendulum
g = acceleration of gravity

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* S. Timoshenko, "Vibration Problems in Engineering," p. 173, D. Van Nostrand Company, Inc., Princeton, N.J., 1955.

Suppose that we consider the case where the length decreases exponentially. That is, suppose that $l = l_0 e^{-\alpha t}$. Substituting this expression into Eq. (2-42), we obtain

$$\ddot{\theta} - 2\alpha\dot{\theta} + \frac{g}{l_0} e^{\alpha t} \sin \theta = 0 \tag{2-43}$$

Solving for the highest derivative, we obtain

$$\ddot{\theta} = 2\alpha\dot{\theta} - \frac{g}{l_0} e^{\alpha t} \sin \theta \tag{2-44}$$

To select scale factors, we note first that solutions are of interest only when $\theta < \pi$ radians; therefore, we use a scale factor of 200 for θ. Also it is noted that the damping coefficient is negative; hence it is to be expected that the resulting oscillations will be divergent. As time increases, therefore, θ and $\dot{\theta}$ will continually increase and the computer will overload. An estimation of $\dot{\theta}$ can be made from analogy to the simple pendulum, but this leads to the maximum value of $\dot{\theta}$ being proportional to the frequency of oscillation. Since the frequency of oscillation will increase with time because the length of the pendulum is shortened, the maximum value of $\dot{\theta}$ will increase with time. Therefore, the scale factor for $\dot{\theta}$ should be selected to be as small as practical. For our purposes we shall select a scale factor of $10\dot{\theta}$. In terms of the scale factors the equation becomes

$$10\ddot{\theta} = 20\alpha\dot{\theta} - 10\frac{g}{l_0} e^{\alpha t} \sin \theta \tag{2-45}$$

The only new feature involved in programming this equation is the generation of the sin θ and multiplication of sin θ by the quantity $(g/l_0)e^{\alpha t}$. This is accomplished by the use of a servo resolver.

The function $(g/l_0)e^{\alpha t}$ is generated in an interesting manner. To do this, we let $y = (g/l_0)e^{\alpha t}$ and form the differential equation whose solution is y. Differentiating y with respect to t, we find the differential equation

$$\frac{dy}{dt} - \alpha y = 0 \tag{2-46}$$

This equation has a very simple circuit consisting of amplifiers 4 and 5 of Fig. 2-22. The magnitude of y is determined by the initial condition on the integrator 4.

To program the resolver, it is necessary to note that the input to the resolver must be expressed in degrees. The scaling in the resolvers available is not uniform, with 1 deg/volt and ½ deg/volt common values. For our purposes we shall assume that a resolver is available which accepts a scale factor of ½ deg/volt. If A volts is impressed across the sine-cosine potentiometer and ±100 volts used on the follow-up potenti-

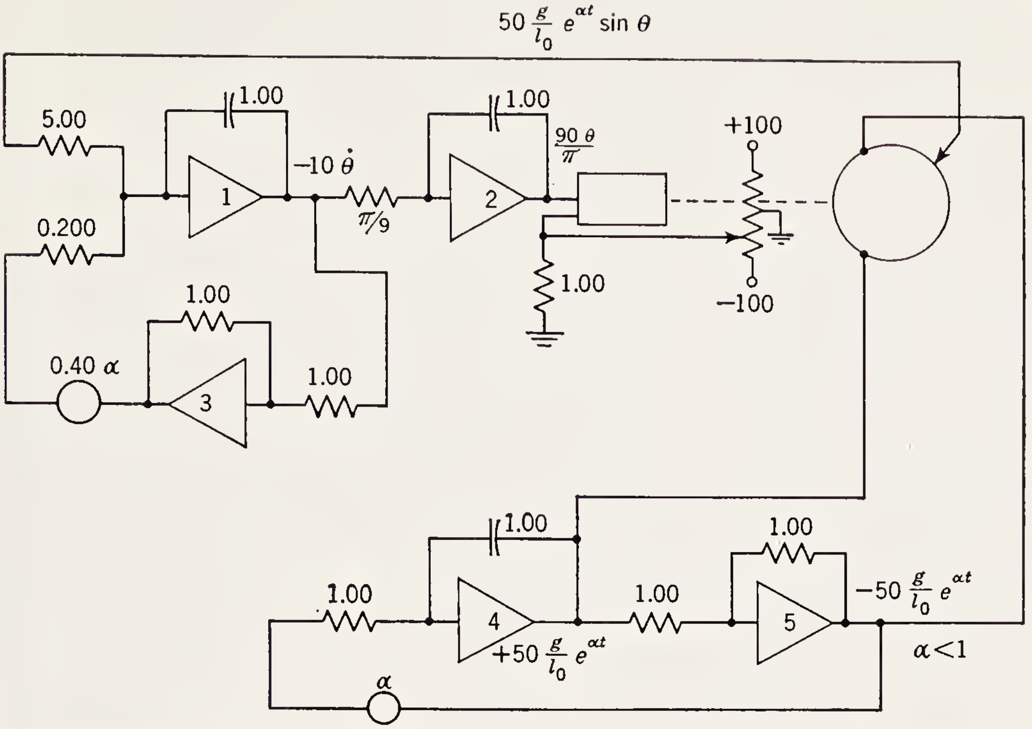

FIG. 2-22. Circuit diagram for solution of the equation $\ddot\theta - 2\alpha\dot\theta + (g/l_0)e^{\alpha t}\sin\theta = 0$.

ometer and an angle of $\theta/2$ deg drives the resolver, then the output from the resolver wiper arm is $A \sin\theta$. In our case $A = 50(g/l_0)e^{\alpha t}$, so that the wiper arm reads $50(g/l_0)e^{\alpha t}\sin\theta$.

2-15. PROGRAMMING A PROBLEM INVOLVING SIMULTANEOUS NONLINEAR DIFFERENTIAL EQUATIONS

Equations (2-47) describe the motion of a mathematical pendulum of length l and mass m.

$$\begin{aligned} m\ddot{x} + ml\ddot\theta\cos\theta - ml\dot\theta^2\sin\theta &= -kx \\ m\ddot{x}l\cos\theta + ml^2\ddot\theta &= -mgl\sin\theta \end{aligned} \tag{2-47}$$

The pendulum is restrained to a fixed base by springs (see Fig. 2-23) which develop a restoring force proportional to the displacement x.

Let us consider the case where

$$m = \frac{w}{g} = 5.00 \text{ slugs}$$
$$l = 3 \text{ ft}$$
$$k = 100 \text{ lb/ft}$$

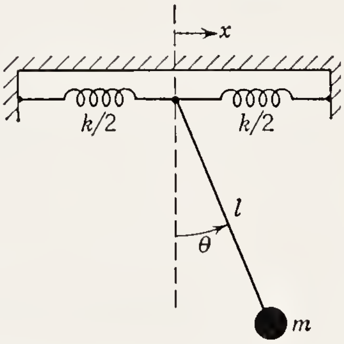

FIG. 2-23. Pendulum with elastic support.

We are interested in obtaining solutions of Eqs. (2-47) for $\theta < \pi/2$.

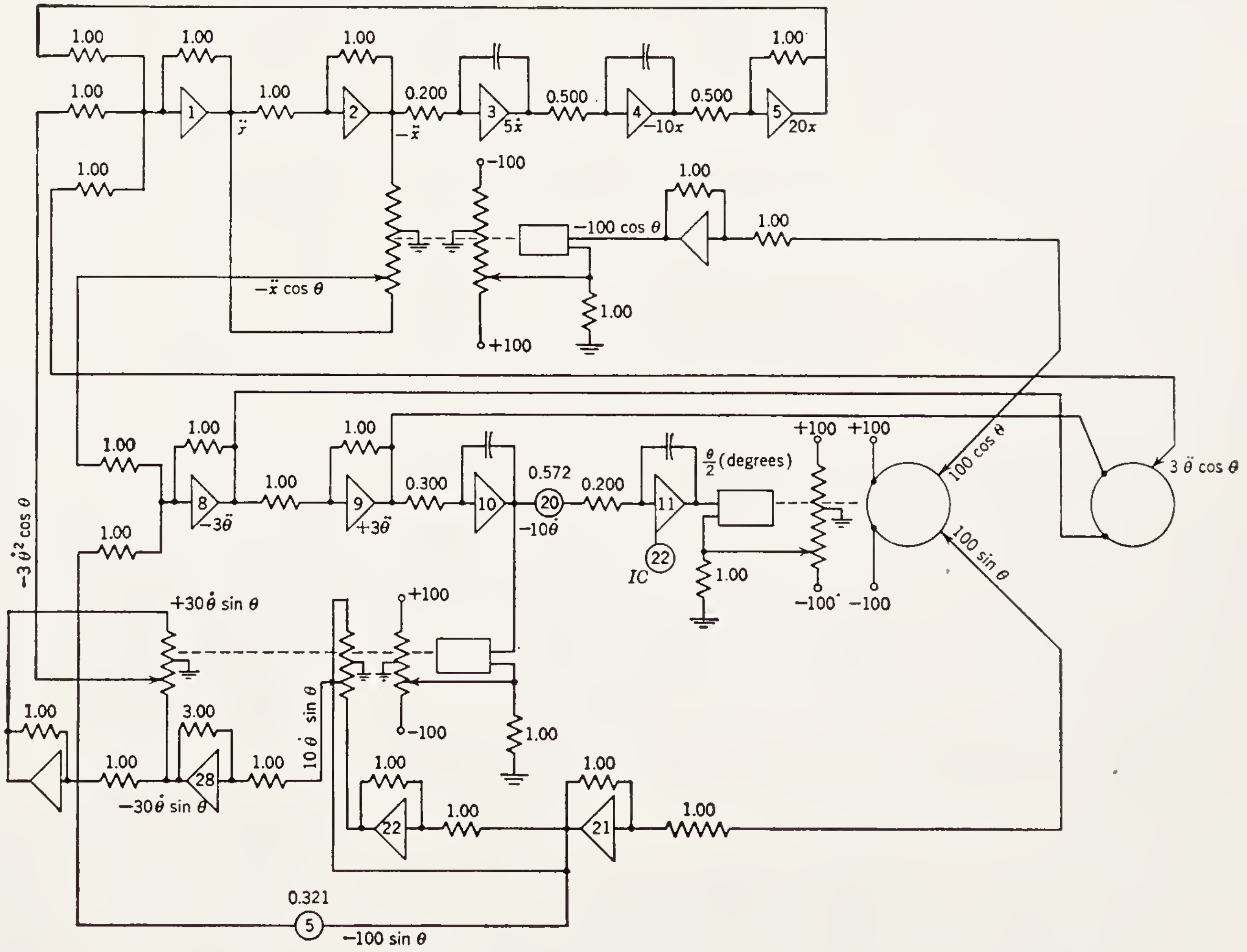

FIG. 2-24. Circuit diagram for pendulum with elastic support.

Substituting in the numerical values and solving for the highest derivatives, we obtain the set of equations

$$\begin{aligned} \ddot{x} &= -20x - 3\ddot{\theta}\cos\theta + 3\dot{\theta}^2\sin\theta \\ \ddot{\theta} &= -\frac{\ddot{x}}{3}\cos\theta - 10.7\sin\theta \end{aligned} \tag{2-48}$$

To select scale factors, we use the fact that $\theta < \pi/2$. Therefore, if we compute θ in degrees and use a servo resolver, we can use a scale factor of ½ volt equals 1 deg. It is not possible by any simple means to determine the maximum range of the remaining variable. Therefore we assume the following scale factors: 20 for x, 5 for x, 1 for x, 10 for $\dot{\theta}$, and 3 for $\dot{\theta}$. The scaled equations are obtained by multiplying the second of Eqs. (2-48) by 3. The programming of these equations offers no new difficulties over those previously described. Figure 2-24 shows a final circuit for solution of Eqs. (2-48). In this circuit the loading of servomultiplier 2 has been set to compensate for feeding input resistors different from 1 megohm.

PROBLEMS

2-1. Set up a circuit for the solution of the third-order differential equation

$$\frac{d^3y}{dt^3} - 13\frac{dy}{dt} + 12y = f(t)$$

where $f(t)$ is a unit step function.

2-2. Set up a circuit to solve the differential equation

$$\ddot{y}(t) + a^2y(t) = e^{\alpha t}$$

for $t \leq 40$, where $y(0) = 0$, $\dot{y}(0) = 50$, $0 \leq \alpha \leq 0.1$, and $0 \leq a \leq 10$.

2-3. Develop a computer circuit for computing the following transfer function:

$$\frac{\theta_o}{\theta_i} = \frac{100 + 39s + 16s^2 + 8s^3}{200 + 112s + 74s^2 + 30s^3}$$

Make a time transformation if necessary.

2-4. Using only one servomultiplier and two amplifiers and given the inputs $\pm\alpha$ and $\pm\beta$, set up a circuit for computing $+\alpha\beta/100 + \beta|\beta|/150 + \beta^3/10{,}000$. Be sure that the loading on all servo potentiometers is the same.

2-5. It is required to generate the error function $y = e^{-x^2/a}$. Differentiate this equation to obtain a differential equation whose solution is $y = e^{-x^2/a}$. Then, assuming that dx/dt is available, prepare a circuit for generation of $y = e^{-x^2/a}$.

2-6. Given $d\phi/dt$, set up a circuit that will give a continuous solution for $\sin\phi$ and $\cos\phi$. *Hint:* Use the relationship that

$$\int \cos\phi \frac{d\phi}{dt}\,dt = \sin\phi$$

and

$$\int \sin\phi \frac{d\phi}{dt}\,dt = -\cos\phi$$

2-7. The equations of motion of two masses elastically coupled and each acted on by a viscous damper are

$$m \frac{d^2x_1}{dt^2} + \beta_1 \frac{dx_1}{dt} + k_1x_1 + k_{12}(x_1 - x_2) = F_1$$

$$m_2 \frac{d^2x_2}{dt^2} + \beta_2 \frac{dx_2}{dt} + k_2x_2 + k_{12}(x_2 - x_1) = 0$$

Let $k_1 = 200$ lb/in. $= 2{,}400$ lb/ft
$k_2 = 230$ lb/in.
$k_{12} = 350$ lb/in.
$m_1 = 4$ slugs
$m_2 = 5$ slugs
$\beta_1 = 32$ lb sec/ft
$\beta_2 = 40$ lb sec/ft

and let F_1, the driving force, be a step function of 275 lb magnitude. Develop a computer diagram for simulating the above mechanical system. Make a time transformation if you feel that it is necessary.

2-8. Given p, q, and r, set up a computer circuit to compute g_x, g_y, and g_z if the rates of these quantities satisfy the following equations:

$$\bar{g} \times \bar{\omega} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ g_x & g_y & g_z \\ p & q & r \end{vmatrix} = \bar{i}\dot{g}_x + \bar{j}\dot{g}_y + \bar{k}\dot{g}_z$$

or

$$\begin{aligned} \dot{g}_x &= g_yr - g_zq \\ \dot{g}_y &= g_zp - g_xr \\ \dot{g}_z &= g_xq - g_yp \end{aligned}$$

Assume that

$$g = \sqrt{g_x^2 + g_y^2 + g_z^2} = 32.2 \text{ ft/sec}$$

and that p, q, and r are less than 2 radians/sec.

2-9. Set up a circuit to solve Mathieu's equation $d^2y/dt^2 + w_0^2(1 + \lambda \cos w_mt)y = 0$, where λ, w_0, and w_m are constant parameters. You are given that

$0 < w_0 \leq 50$
$0 < w_m \leq 100$
$0 < \lambda \leq 1$

2-10. The differential equation

$$\frac{\ddot{q}}{K^2} + \frac{2D\dot{q}}{K} + q + \tfrac{4}{3}q^3 = S \cos \omega t$$

is known as Duffing's equation and occurs in the study of superharmonic oscillations.*

Let D = damping ratio = 0.02
S = amplitude of forcing function = 1.0, 1.5, 2.0
ω = frequency range of forcing function < 1.0
K = natural frequency of linear system = 0.75, 1.00

Obtain the forcing function as a solution to an appropriate differential equation. Set up a circuit to solve Duffing's equation which is adjustable over the range of the parameters given.

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* C. P. Atkinson, Superharmonic Oscillations as Solutions to Duffing's Equation as Solved by an Electronic Differential Analyzer, *J. Appl. Mechanics*, December, 1957.

2-11. The equation

$$3\lambda \frac{d^3y}{dt^3} + y\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 = 0$$

subject to the conditions $y = \ddot{y} = 0$ when $t = 0$, $\dot{y} \rightarrow 0$ when $t \rightarrow \infty$, occurs in the theory of the plane jet in hydrodynamics. Set up a circuit to solve this equation for y for arbitrary λ.

2-12. Set up a circuit to solve the differential equation

$$\frac{d^2y}{dt^2} - \frac{1}{2}(1 + 12y^2) = 0$$

Make estimates of the magnitudes of the problem variables, and determine suitable scale factors, for various initial values of $\dot{y}$ and y.

2-13. Consider a gravity pendulum having a bearing with a horizontal axis. In addition to the usual swinging motion about the horizontal bearing, the horizontal bearing is rotated at constant angular velocity ω about a vertical axis. The equation of motion for such a pendulum is

$$\ddot{\theta} + \frac{g}{\lambda}\sin\theta\left(\lambda - \frac{\omega^2\lambda}{g}\cos\theta\right) = 0$$

Set up a circuit for solution of this equation for $g/\lambda = 10$ and $\omega^2\lambda/g = 1, 2, 3, 4, 5$.

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6. Howe, R. M.: "Scaling for Electronic Differential Analyzers," report, University of Michigan, Department of Aeronautical Engineering, Ann Arbor, Mich.

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CHAPTER 3

Simulation of Linear Systems

3-1. INTRODUCTION

The preceding chapters have familiarized the reader with the basic components of the analog computer and with the elementary techniques by means of which these components are applied to the solution of physical problems. In this chapter we shall consider the analysis of linear systems* on an analog computer in much greater detail than was done in Chap. 2. Certain standard techniques for the analysis of linear systems are presented for those readers whose experience is limited. Notably, block-diagram notation is explained briefly, and a section on the preparation of Bode plots is included.

The programming of analog solutions for linear differential equations can be grouped into two sets: those arising from the direct solution of the differential equation and those utilizing network methods. Both general techniques are considered and a comparison of the two methods made.

Among the more interesting problems involving linear systems are the approximation and simulation of dead time. Some of the more useful methods by which an analog computer can be employed to perform this function will be considered in detail, while others will be mentioned only briefly.

One of the operations most commonly performed on an analog computer is the simulation of linear transfer functions. For this reason, many analog-computer techniques have been developed with this specific goal in mind. These techniques are not always completely applicable to the solution of more general linear differential equations. For example, in simulating a linear system involving initial conditions, care must be taken to be sure the method of solution chosen allows for the insertion of these conditions. This is particularly true for the case of network methods, where modification is usually necessary to insert the initial conditions (see Sec. 3-6, Example 3).

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* Throughout this chapter, the term linear systems will be used to mean linear time-invariant systems.

3-2. BLOCK-DIAGRAM NOTATION

In the analysis of servomechanisms it is convenient to use a graphical representation to describe a system. This is done by letting a "box" or "block" represent each element of the system. With this block is then associated the transfer function of the element. For example, consider Fig. 3-1, which shows the block diagram for a simple feedback system.

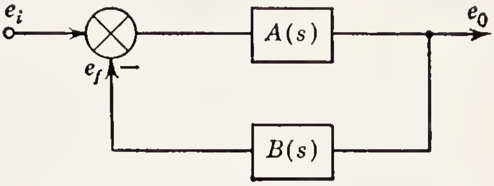

FIG. 3-1. A simple block diagram.

The symbol ⊕ is used to indicate a summing junction, the arrowheads indicate inputs to the summing junction, and the minus sign next to the one arrowhead indicates that that term is to be summed in a negative sense, i.e., subtracted. The path containing the element $A(s)$ is called the forward path, or loop; that containing the element $B(s)$ is called the backward, or feedback, loop. From analysis of the block diagram, it is possible to write the transfer function of the entire system in terms of the transfer functions of the elements. Thus, since the input to the element 1 is $\epsilon = E_i - E_f$ and the transfer function of this element is $A(s)$, we can write

$$\frac{E_o}{E_i - E_f} = A(s) \qquad \text{or} \qquad E_o = A(s)(E_i - E_f) \tag{3-1}$$

In the same manner we can write

$$E_f = B(s)E_o \tag{3-2}$$

Combining Eqs. (3-1) and (3-2) and solving for the ratio E_o/E_i, we have

$$\frac{E_o}{E_i} = \frac{A(s)}{1 + A(s)B(s)} \tag{3-3}$$

A second example will show further how the application of this technique simplifies the analysis of complex systems. Consider the multiloop system whose block diagram is shown in Fig. 3-2.

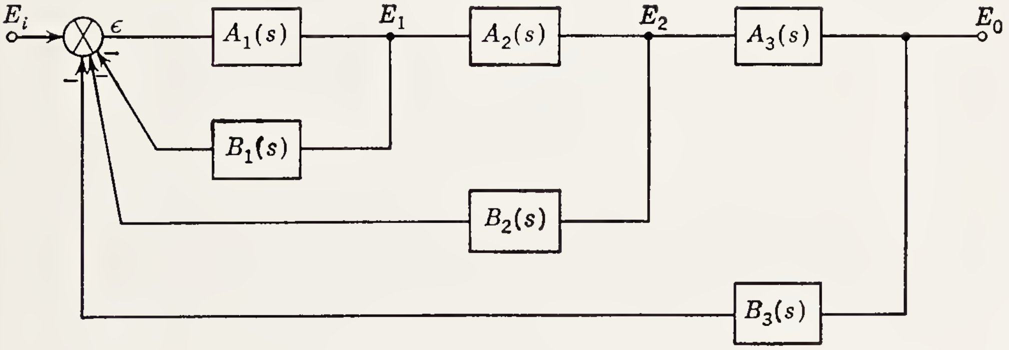

FIG. 3-2. Block diagram for a system having three feedback loops.

From this diagram we can write the following equations for the forward path:

$$\begin{aligned} E_1 &= A_1(s)\epsilon \\ E_2 &= A_1(s)A_2(s)\epsilon \\ E_o &= A_1(s)A_2(s)A_3(s)\epsilon \end{aligned} \tag{3-4}$$

but ϵ can be expressed

$$\epsilon = E_i - B_1(s)E_1 - B_2(s)E_2 - B_3(s)E_o \tag{3-5}$$

Therefore, on substituting for E_o, E_1, and E_2 and solving for the ratio ϵ/E_i, we obtain

$$\frac{\epsilon}{E_i} = \frac{1}{1 + A_1(s)B_1(s) + A_1(s)A_2(s)B_2(s) + A_1(s)A_2(s)A_3(s)B_3(s)} \tag{3-6}$$

Thus, on eliminating ϵ we can write the transfer function of the system as

$$\frac{E_o}{E_i} = \frac{A_1(s)A_2(s)A_3(s)}{1 + A_1(s)B_1(s) + A_1(s)A_2(s)B_2(s) + A_1(s)A_2(s)A_3(s)B_3(s)} \tag{3-7}$$

3-3. THEORY AND APPLICATION OF BODE PLOTS

Since a linear system can be described completely in terms of its response to a sinusoidal input,* the analysis of the so-called frequency response becomes of great importance. Because practically every application of the analog computer involves at least some linear elements, it is highly desirable that every analog-computer operator should at least be familiar with some of the basic principles governing the frequency response of linear systems. This is especially true for those operations in which the programmer is expected to take raw data about a given system and convert them into a set of differential equations describing that system. For example, if the behavior of a system is linear or nearly so, the response to sinusoidal inputs may be actually measured. Once this has been done, it is necessary to match a transfer function to this frequency response. To facilitate this matching, it is convenient to make graphs of phase angle and amplitude ratio versus frequency. These graphs generally bear the name Bode plots. The term Bode plots as used in this text will be restricted to these phase and attenuation diagrams. The pertinent features of Bode plots may be simply summarized in the following manner:

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* Ref. 3, chap. 3.

The transfer function of any linear system can be factored into the form

$$\frac{E_o}{E_i} = \frac{Ks^n \prod_{i=1}^{n_1} (1 + T_i s) \prod_{k=1}^{n_2} \left(1 + \frac{2\xi_k}{\omega_k} s + \frac{s^2}{\omega_k{}^2}\right)}{\prod_{l=1}^{n_3} (1 + T_l s) \prod_{m=1}^{n_4} \left(1 + \frac{2\xi_m}{\omega_m} s + \frac{s^2}{\omega_m{}^2}\right)} \tag{3-8}$$

where K, n, T_l, ξ_k, T_i, and ξ_m can take on either positive or negative values. For example, the transfer function

$$\frac{E_o}{E_i} = \frac{1 - 0.5s + 0.99s^2 + 0.005s^3 - 0.01s^4}{s^2 + 1.05s^3 + 1.05s^4 + 0.05s^5} \tag{3-9}$$

can be written

$$\frac{E_o}{E_i} = \frac{(1 + 0.1s)(1 - 0.1s)(1 - 0.5s + s^2)}{s^2(1 + 0.05s)(1 + s + s^2)} \tag{3-10}$$

If $s = j\omega$ is substituted in Eq. (3-8), the frequency response is obtained directly. Thus from Eq. (3-8) we have

$$\left|\frac{E_o}{E_i}\right| = \frac{|K|\ |j\omega|^n \prod_{i=1}^{n_1} |1 + T_i j\omega| \prod_{k=1}^{n_2} \left|1 + \frac{2\xi_k}{\omega_k} j\omega - \frac{\omega^2}{\omega_k{}^2}\right|}{\prod_{l=1}^{n_3} |1 + T_l j\omega| \prod_{m=1}^{n_4} \left|1 + \frac{2\xi_m}{\omega_m} j\omega - \frac{\omega^2}{\omega_m{}^2}\right|} \tag{3-11}$$

Therefore, we have that the amplitude ratio of the total system is the product and/or quotient of terms of the form $|K|$, $|u|$, $|1 + ju|$, and $|1 \pm 2j\xi u - u^2|$, where $u = \omega/\omega_n$; $n = i, k, l, m$; and $\omega_{i,k} = 1/T_{i,l}$. These parameters are assigned the following names:

$T_{i,l}$ = time constant

ω_n = undamped natural, or break, frequency

$\xi_{k,m}$ = damping ratio $= \dfrac{\text{damping present}}{\text{critical damping}}$

u = nondimensional frequency $= \dfrac{\omega}{\omega_n}$

Now, if we deal with the logarithm of the amplitude ratio rather than the ratio itself, the products of Eq. (3-11) reduce to summations of the logarithms of the components. It is convenient to introduce the unit of the decibel, which is defined by the relationship

$$\left|\frac{E_o}{E_i}\right|_{\text{db}} = 20 \log_{10} \left|\frac{E_o}{E_i}\right| \tag{3-12}$$

and is abbreviated db. Thus, since the amplitude ratio measured in decibels can be expressed as the sum or difference of terms of the form

$$\begin{aligned} &20 \log |K|,\ 20 \log |u| \\ &20 \log |1 + ju|,\ 20 \log |1 + 2\xi ju - u^2| \end{aligned}$$

it suffices to analyze the behavior of the individual terms.*

From elementary complex-variable theory we know that any complex variable Z can be written $Z = re^{i\theta}$, where $\theta = \arctan (\text{Im } Z/\text{Re } Z)$. Employing this relationship, we can write

$$Z_1 Z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

and

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Thus from these equations we can see that the total phase angle ϕ of Eq. (3-11) can be written as the sum of the phase angles of the factors of the numerator minus the sum of the phase angles of the factors of the denominator.

Thus the analysis of the transfer function of any linear system reduces to the study of the amplitude ratio and phase angle of the four types of factors of Eq. (3-8). We shall therefore consider each of these types of factor in detail.

The gain factor K, assuming K real, has the amplitude ratio

$$\left|\frac{E_o}{E_i}\right|_{\text{db}} = 20 \log |K|$$

and a zero phase angle. Thus, the gain factor merely translates the zero decibel reference of the ordinate of Eq. (3-11).

The factor ju has the amplitude ratio

$$\left|\frac{E_o}{E_i}\right|_{\text{db}} = 20 \log |u|$$

which is a straight line versus frequency with a slope of 6 db/octave (where an octave is defined as the interval between two frequencies having the ratio 2). This line has a zero ordinate when $u = 1$. The phase angle of this term is given by

$$\phi = \arctan \frac{u}{0} = \arctan \infty = 90 \text{ deg}$$

which is a constant for all frequencies.

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* Unless otherwise noted, all logarithms in this section are taken to the base 10.

The term $1 + ju$ has the amplitude ratio

$$\left|\frac{E_o}{E_i}\right|_{\text{db}} = 20 \log \sqrt{1 + u^2} \tag{3-13}$$

and the phase angle

$$\phi = \arctan u$$

Neither of these equations is as simple as those for the previous cases. However, the amplitude ratio may be plotted rapidly by determining the asymptotes to the curve at high and low frequencies. The deviation from the asymptotes in the middle-frequency range may be determined by simple calculations. Equation (3-13) can be written

$$\left|\frac{E_o}{E_i}\right|_{\text{db}} = \frac{10}{2.3} \log_e (1 + u^2) = \frac{10}{2.3}\left(u^2 - \frac{u^4}{2} + \frac{u^6}{3} - \cdots\right)$$

For small values of u this is an alternating series of decreasing terms; hence

$$\left|\frac{E_o}{E_i}\right|_{\text{db}} < \frac{10}{2.3} u^2$$

Thus, if $u < 0.1$, $|E_o/E_i|_{\text{db}} < 0.043$ db and we can see that the amplitude ratio approaches 0 db asymptotically. To facilitate determination of the high-frequency asymptote, we rewrite Eq. (3-13) as

$$\left|\frac{E_o}{E_i}\right|_{\text{db}} = 20\left[\log u + \log\left(1 + \frac{1}{u^2}\right)^{1/2}\right] \tag{3-14}$$

As $u \rightarrow \infty$, $\log (1 + 1/u^2)^{1/2} \rightarrow 0$; so we have that

$$\left|\frac{E_o}{E_i}\right|_{\text{db}} \rightarrow 20 \log u \qquad \text{as } u \rightarrow \infty$$

Hence, the high-frequency asymptote has a slope of 6 db/octave. The two asymptotes intersect at $u = 1$. The term break frequency has reference to the sharp slope change in the asymptotes at the point of intersection. The following calculations will establish the difference between the true curve and the asymptotes:

At $u = 0.50$: $\left|\frac{E_o}{E_i}\right|_{\text{db}} = 20 \log \sqrt{1.25} = 1$ db

At $u = 1.00$: $\left|\frac{E_o}{E_i}\right|_{\text{db}} = 20 \log \sqrt{2} = 3$ db

At $u = 2.00$: $\left|\frac{E_o}{E_i}\right|_{\text{db}} = 20 \log \sqrt{5} = 7$ db

Therefore, with respect to the asymptotes, the true curve is 1 db below the asymptotes at one-half (one octave below) and twice (one octave above) the break frequency and is 3 db below the asymptotes at the break frequency. The phase function is not readily approximated by asymptotic methods but can be determined easily from a table of trigonometric functions. For convenience in making phase-attenuation plots Fig. 3-3 presents a plot of amplitude ratio and phase angle versus nondimensional frequency. It should be remembered that, at the break frequency $u = 1$, the phase angle for a simple first-order lag is 45 deg.

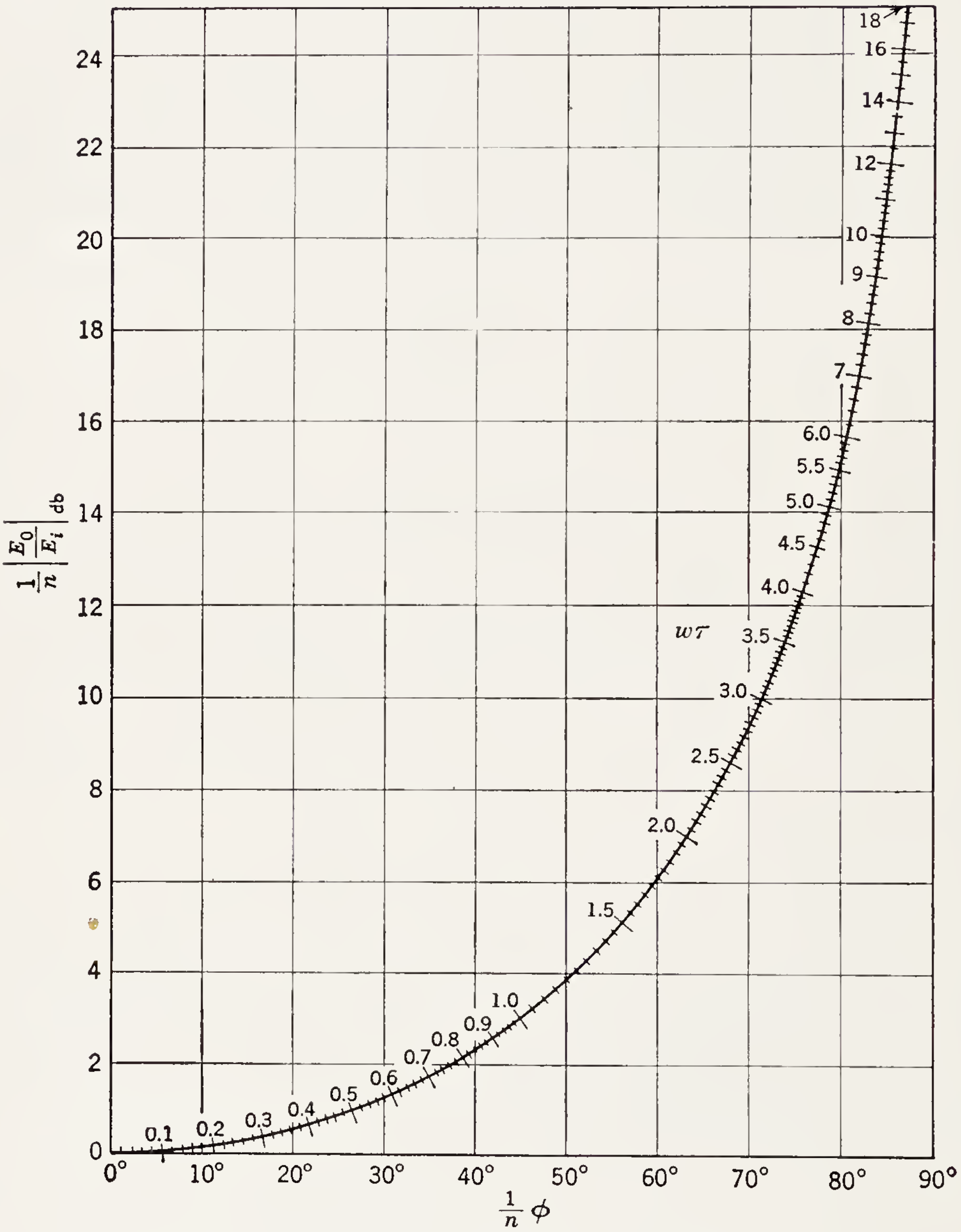


FIG. 3-3. Phase-attenuation characteristic of a first-order lag system.

The quadratic factor $1 + 2ju - u^2$ has the amplitude ratio

$$\left|\frac{E_o}{E_i}\right|_{\mathrm{db}} = 20 \log \sqrt{(1 - u^2)^2 + 4\xi^2u^2}$$

and the phase angle

$$\phi = \arctan \frac{2\xi u}{1 - u^2}$$

Using similar techniques to those employed for the first-order factor, it can be shown that the low-frequency asymptote is again zero and that the high-frequency asymptote is given by

$$\left|\frac{E_o}{E_i}\right|_{\mathrm{db}} = 20 \log u^2 = 40 \log u$$

Thus the asymptotes for the quadratic case are similar to those for the first-order factor with the exception that the high-frequency asymptote has twice the slope. The deviation from the asymptote, however, is a function of the damping ratio ξ and cannot be expressed simply, as was done in the previous case. The phase relationship is also dependent on damping and cannot be expressed simply. For this reason there is presented, as Fig. 3-4, a plot of amplitude ratio and phase angle versus nondimensional frequency. To determine the amplitude ratio and phase angle from this figure, one first finds the intersection of the curve representing the desired value of u and the curve representing the damping ratio ξ of the factor. Once this intersection is determined, the phase angle in degrees is read from the scale on the right margin and the amplitude ratio in decibels is read from the scale on the upper margin. This particularly ingenious method of plotting these functions was conceived by S. H. Lewis, of the Douglas Aircraft Company.

The following example will illustrate the construction of Bode plots.

Example. It is desired to construct a Bode plot for the transfer function

$$\frac{E_o}{E_i} = \frac{2(1 + 0.20s)}{(1 + 0.4s)(1 + 0.10s + 0.01s^2)} \tag{3-15}$$

The first step toward performing this operation is to break the function down into its four factors, i.e.,

$$2,\ 1 + 0.20s,\ \frac{1}{1 + 0.4s},\ \frac{1}{1 + 0.10s + 0.01s^2}$$

Considering the amplitude ratio, we next plot the asymptotes for each factor. The sum of the individual asymptotes is the asymptote of the

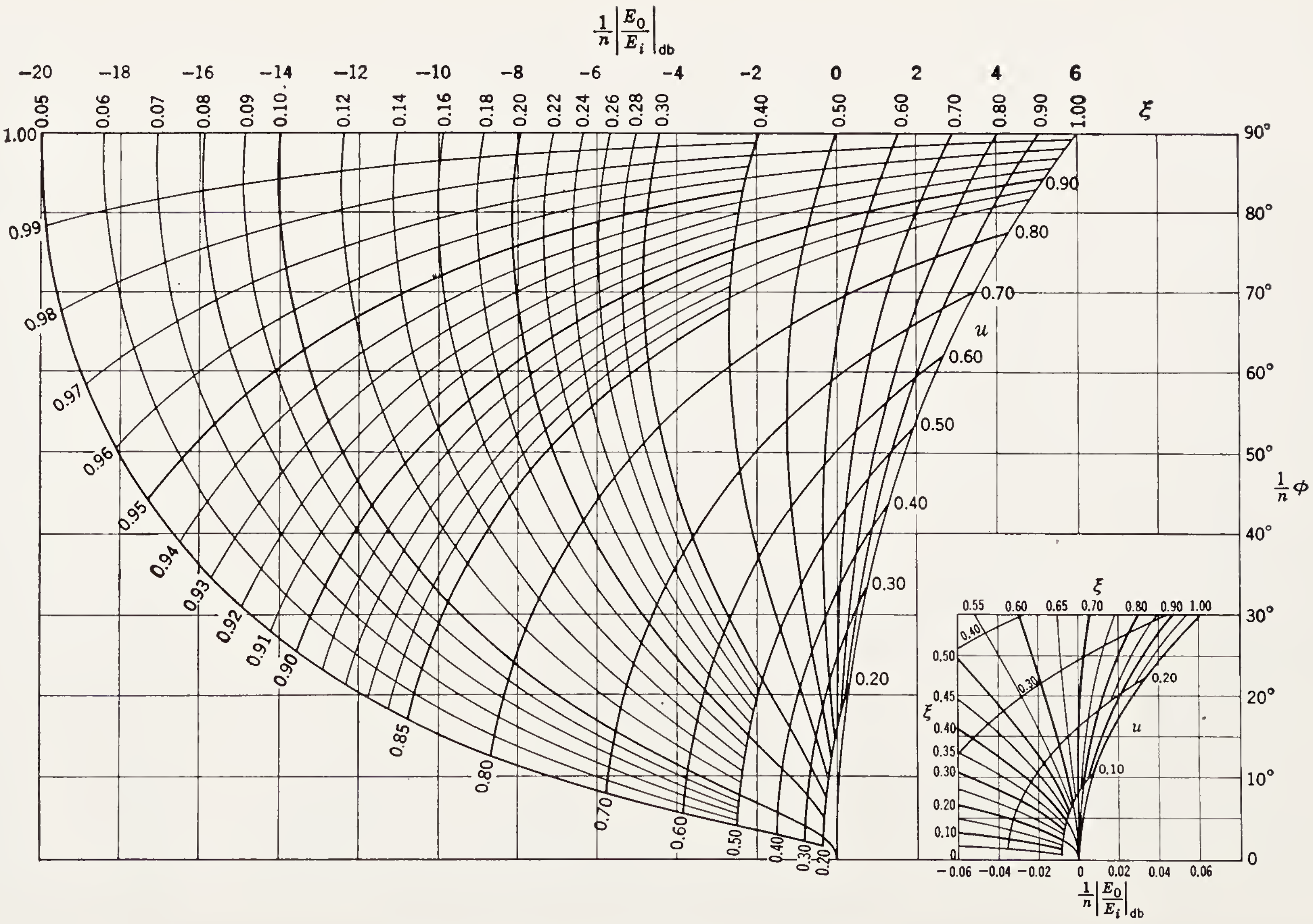


FIG. 3-4 Second-order Bode characteristics.

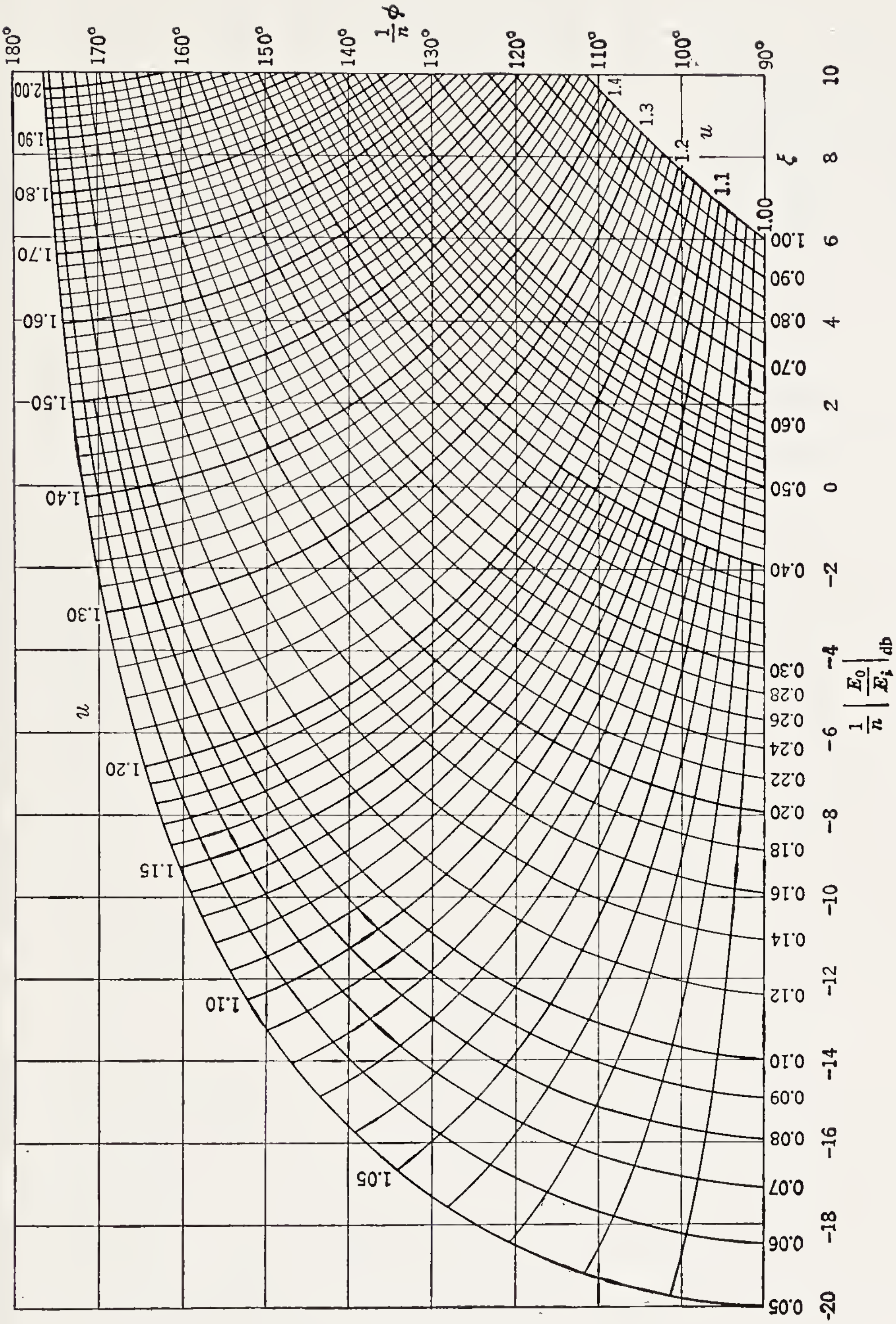


FIG. 3-4 (*Continued*)

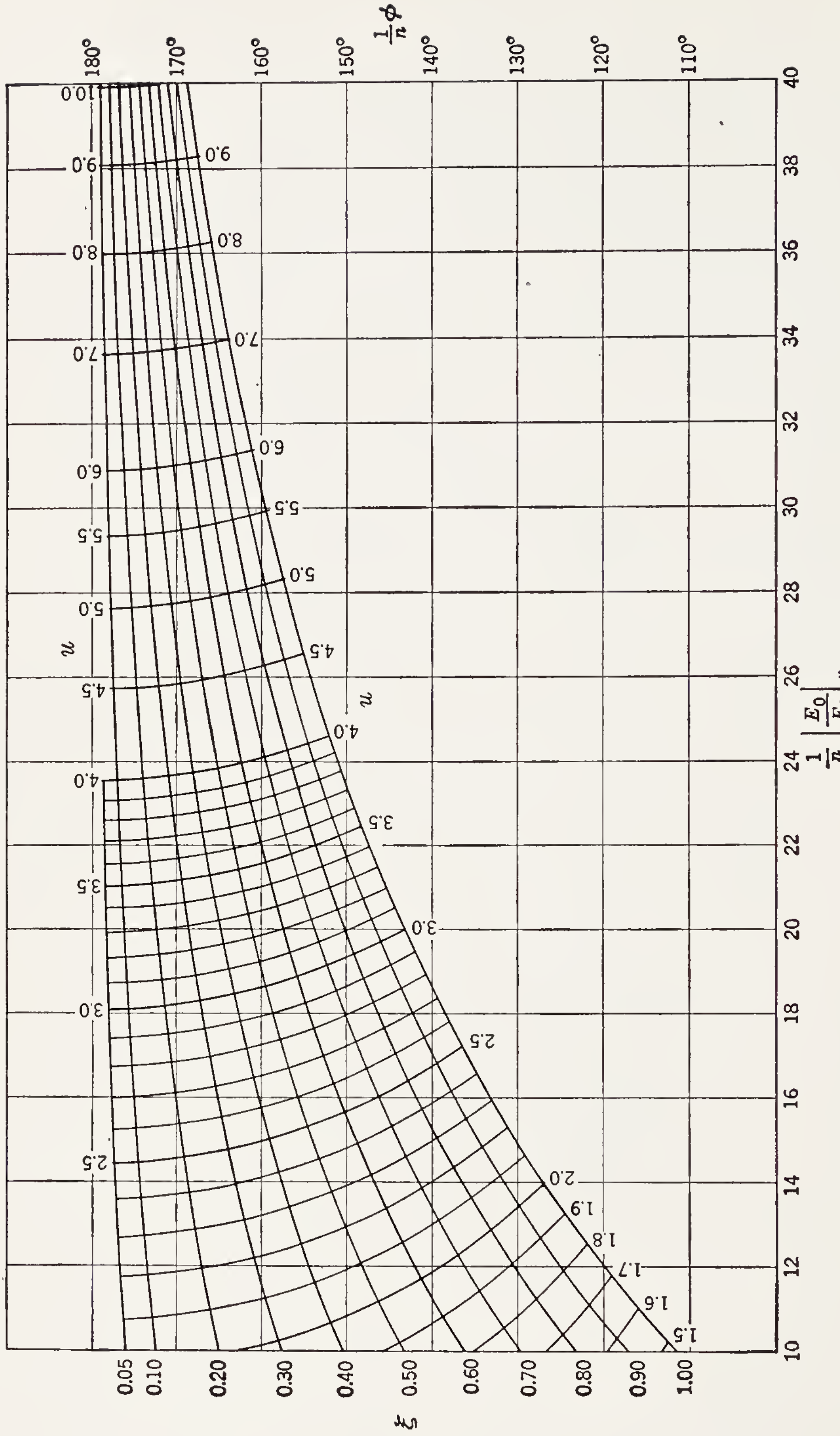


FIG. 3-4. *(Continued)*

total function. From Eq. (3-15) the natural frequencies are

$$\omega_1 = 2.5 \text{ radians/sec}$$
$$\omega_2 = 5.0 \text{ radians/sec}$$
$$\omega = 10.0 \text{ radians/sec}$$

and the damping ratio of the second-order factor is given by $\xi = 0.5$. The asymptotes are shown as dashed lines in Fig. 3-5. It will be noted that the natural frequencies have been converted from radians per second, as they appear in Eq. (3-15), to cycles per second for this plot.

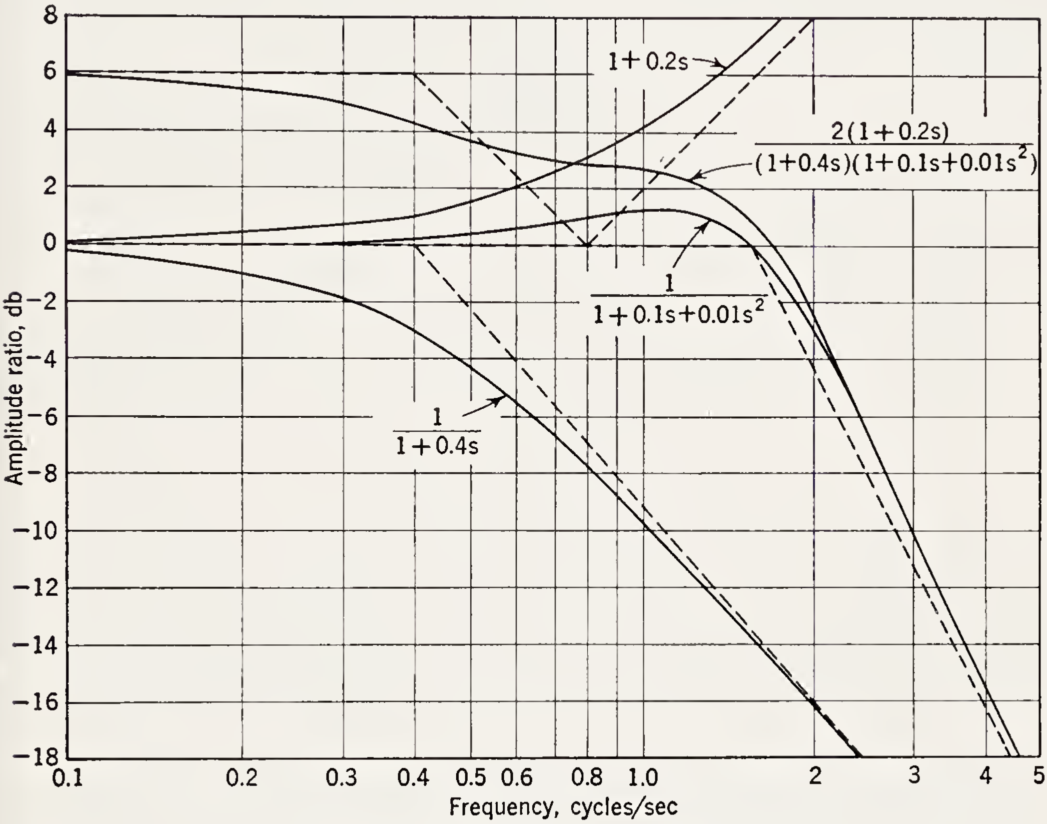


FIG. 3-5. Amplitude characteristic.

Once the asymptotes have been established, the first-order factors can be plotted directly, using the rule given previously. The second-order curve is plotted using Fig. 3-4. These curves are then added graphically to obtain the final curve. The actual amplitude-ratio curves are shown in Fig. 3-5 as solid lines. The phase angles of the individual factors are plotted graphically to give the total phase angle of the transfer function. This operation is illustrated by Fig. 3-6.

The Bode plot can be used to determine an approximate transfer function for any piece of hardware whose characteristics are sufficiently linear. If the response of the item of hardware to a sinusoidal input is measured,

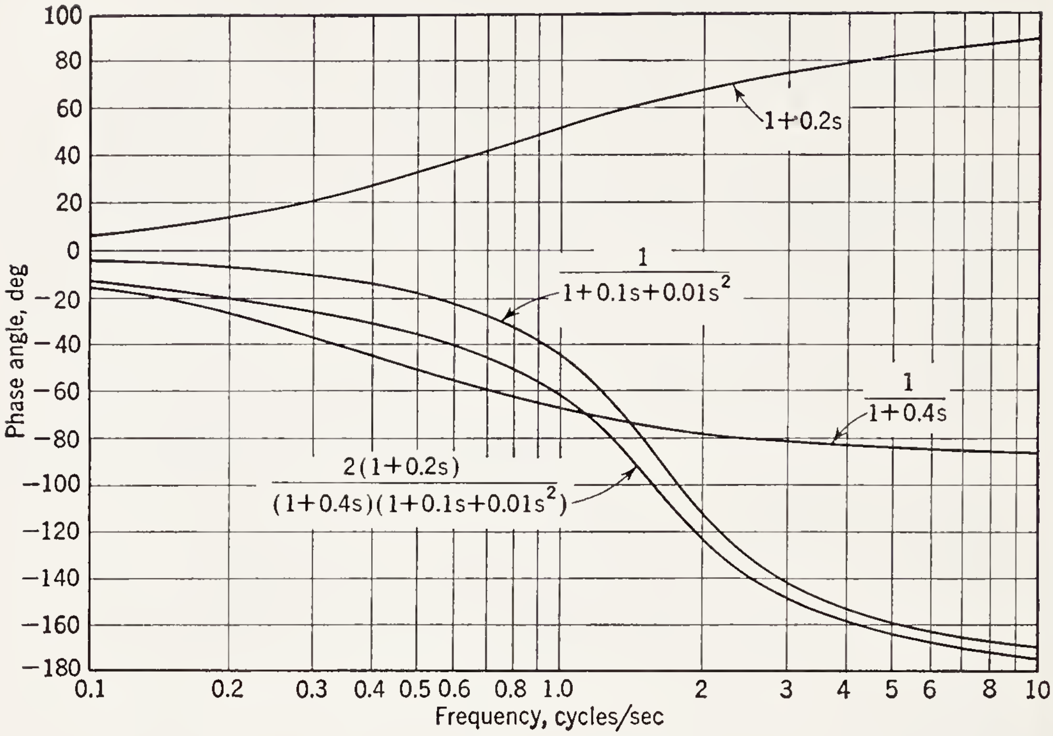


Fig. 3-6. Phase characteristics.

plots of amplitude ratio and phase angle versus frequency can be made. A curve-fitting process is then followed, so that the phase and amplitude curves are fitted with terms of the form of those of Eq. (3-8).* This results in a linear transfer function which approximates the characteristics of the hardware.

3-4. SIMULATION OF LINEAR SYSTEMS BY THE DIRECT SOLUTION OF THE DIFFERENTIAL EQUATIONS

The most obvious method for the analysis of a linear system is the direct solution of the differential equations. This technique has already been used in Chap. 2, where the *M* method of programming was introduced. A slightly different approach to the problem will be described here. A simple example will suffice to describe the technique. Suppose that a servomechanism has the transfer function

$$\frac{E_o}{E_i} = \frac{1 + 0.1s}{(1 + 0.5s)(1 + 0.05s)} = \frac{1 + 0.1s}{1 + 0.55s + 0.025s^2} \tag{3-16}$$

---

* Bode has demonstrated that any arbitrary amplitude and phase can be approximated over a finite frequency range to any desired accuracy. See H. W. Bode, "Network Analysis and Feedback Amplifier Design," D. Van Nostrand Company, Inc., Princeton, N.J., 1945.

Then, converting to a differential equation,

$$e_o + 0.55\dot{e}_o + 0.025\ddot{e}_o = e_i + 0.10\dot{e}_i$$

Integrating with respect to t, we obtain

$$\int e_o\,dt + 0.55e_o + 0.025\dot{e}_o = \int e_i\,dt + 0.10e_i$$

which can be written as

$$\dot{e}_o = 40\int(e_i - e_o)\,dt + 4e_i - 22e_o \tag{3-17}$$

A circuit to solve Eq. (3-17) is given in Fig. 3-7.

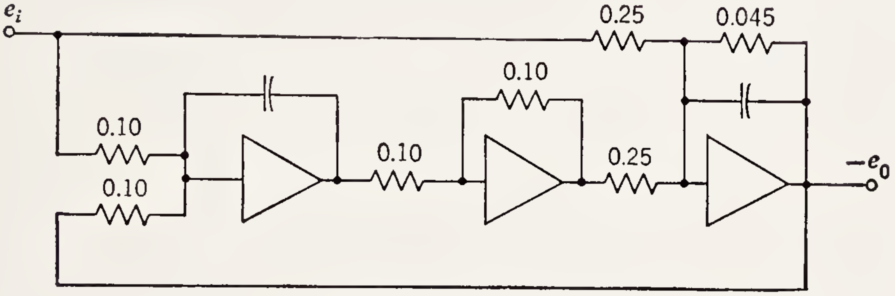


Fig. 3-7. Circuit for

$$\frac{E_o}{E_i} = \frac{1 + 0.1s}{(1 + 0.05s)(1 + 0.5s)}$$

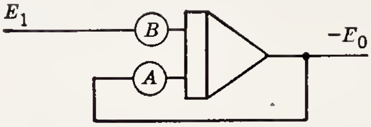
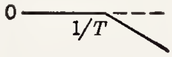
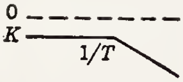
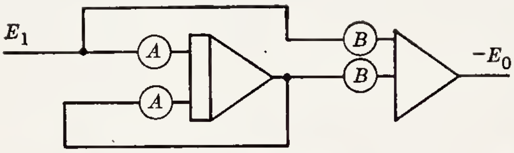
The lack of flexibility in many large-scale computers makes it difficult to use any method other than that outlined above. This is especially true for those computers employing fixed gains and feedbacks. The simulation of transfer functions on such computers is greatly simplified by the use of Table 3-1. This table, which was prepared and distributed by Electronics Associates, Inc., consists of a collection of simple transfer functions and the circuits to simulate them. It also includes the asymptotes of a Bode plot of amplitude ratio versus frequency for convenience in evaluating system performance. The following example will serve to illustrate the use of this table.

Example. It is desired to simulate a linear system having the transfer function

$$\frac{E_o}{E_i} = \frac{\frac{1}{2}(1 + s)}{(1 + 0.5s)(1 + 0.2s)}$$

An examination of Table 3-1 shows that circuit 21 can be used to give this transfer function. It should be noticed that circuit 28 yields the same transfer function, but with the negative sign. The time constants

TABLE 3-1

<table>
<tr><th>No.</th><th>Bode plot</th><th>Transfer function</th><th>Time constants</th><th>Gains</th></tr>
<tr><td colspan="5">
</td></tr>
<tr><td>1</td><td>
</td><td>$\frac{1}{Ts+1}$</td><td>$T = \frac{1}{A}$</td><td>$A = B = \frac{1}{T}$</td></tr>
<tr><td>2</td><td>
</td><td>$\frac{K}{Ts+1}$</td><td>$T = \frac{1}{A}$
$K = \frac{B}{A}$</td><td>$A = \frac{1}{T}$
$B = AK$</td></tr>
<tr><td colspan="5">
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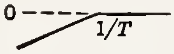
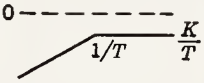
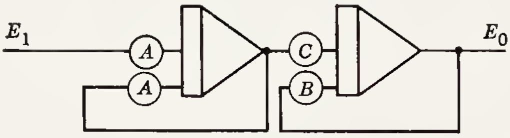
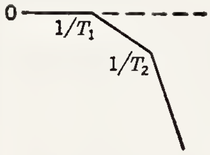
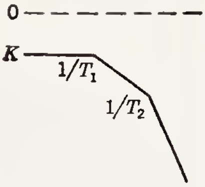
<tr><td>3</td><td>
</td><td>$\frac{Ts}{Ts+1}$</td><td>$T = \frac{1}{A}$</td><td>$A = \frac{1}{T}$
$B = 1$</td></tr>
<tr><td>4</td><td>
</td><td>$\frac{Ks}{Ts+1}$</td><td>$T = \frac{1}{A}$
$K = BT$</td><td>$A = \frac{1}{T}$
$B = AK$</td></tr>
<tr><td colspan="5">
</td></tr>
<tr><td>5</td><td>
</td><td>$\frac{1}{(T_1s+1)(T_2s+1)}$</td><td>$T_1 = \frac{1}{A}$
$T_2 = \frac{1}{B}$</td><td>$A = \frac{1}{T_1}$
$B = \frac{1}{T_2}$
$C = B$</td></tr>
<tr><td>6</td><td>
</td><td>$\frac{K}{(T_1s+1)(T_2s+1)}$</td><td>$T_1 = \frac{1}{A}$
$T_2 = \frac{1}{B}$</td><td>$A = \frac{1}{T_1}$
$B = \frac{1}{T_2}$
$C = \frac{K}{T_2}$</td></tr>
</table>

TABLE 3-1 (*Continued*)

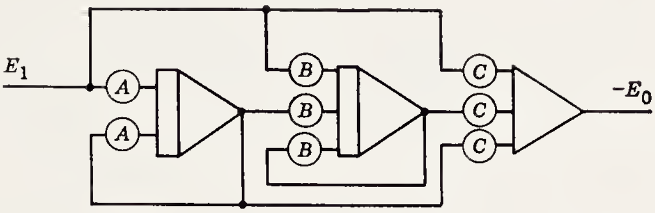
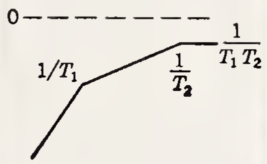
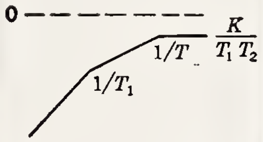
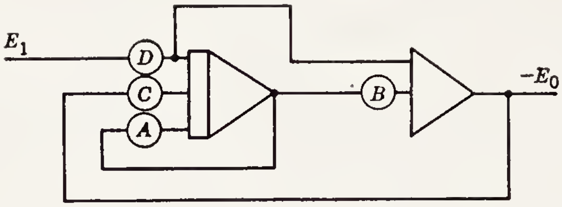
<table>
<tr><th>No.</th><th>Bode plot</th><th>Transfer function</th><th>Time constants</th><th>Gains</th></tr>
<tr><td colspan="5"> </td></tr>
<tr><td>7</td><td> </td><td>$\frac{s^2}{(T_1s+1)(T_2s+1)}$</td><td>$T_1 = \frac{1}{A}$
$T_2 = \frac{1}{B}$</td><td>$A = \frac{1}{T_1}$
$B = \frac{1}{T_2}$
$C = \frac{1}{T_1T_2}$</td></tr>
<tr><td>8</td><td> </td><td>$\frac{Ks^2}{(T_1s+1)(T_2s+1)}$</td><td>$T_1 = \frac{1}{A}$
$T_2 = \frac{1}{B}$
$K = \frac{C}{AB}$</td><td>$A = \frac{1}{T_1}$
$B = \frac{1}{T_2}$
$C = \frac{K}{T_1T_2}$</td></tr>
</table>

TABLE 3-1 (*Continued*)



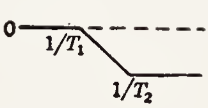
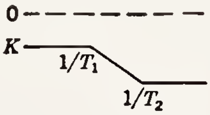
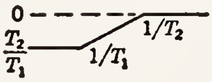
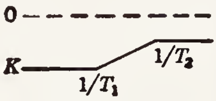
No.	Bode plot	Transfer function	Time constants	Gains
9		$\frac{T_2 s + 1}{T_1 s + 1}$	$T_1 = \frac{1}{A - BC}$ $T_2 = \frac{1}{A - B}$	$A = B + \frac{1}{T_2}$ $C = \frac{T_1 - T_2}{BT_1T_2} + 1$ $D = \frac{T_2}{T_1}$
10		$\frac{K(T_2 s + 1)}{T_1 s + 1}$	$T_1 = \frac{1}{A - BC}$ $T_2 = \frac{1}{A - B}$ $K = \frac{T_1 D}{T_2}$	$A = B + \frac{1}{T_2}$ $C = \frac{T_1 - T_2}{BT_1T_2} + 1$ $D = \frac{KT_2}{T_1}$
11		$\frac{K(T_1 s + 1)}{T_2 s + 1}$	$T_1 = \frac{1}{A - B}$ $T_2 = \frac{1}{A}$ $K = \frac{T_2}{T_1}$	$A = \frac{1}{T_2}$ $B = \frac{T_2 - T_1}{T_1T_2}$ $C = 0$ $D = 1$
12		$\frac{K(T_1 s + 1)}{T_2 s + 1}$	$T_1 = \frac{1}{A - B}$ $T_2 = \frac{1}{A}$ $K = \frac{T_2 D}{T_1}$	$A = \frac{1}{T_2}$ $B = \frac{T_2 - T_1}{T_1T_2}$ $C = 0$ $D = \frac{KT_1}{T_2}$

TABLE 3-1 (*Continued*)

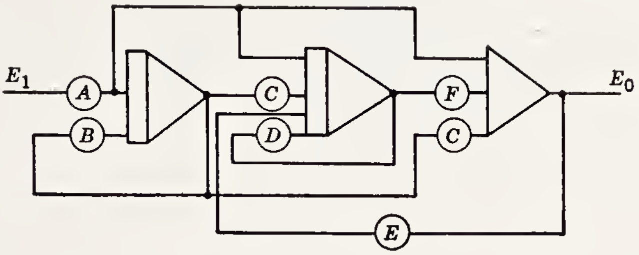
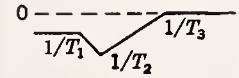
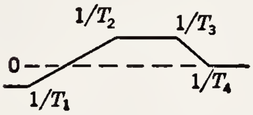
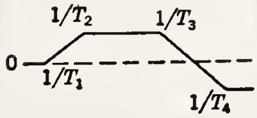
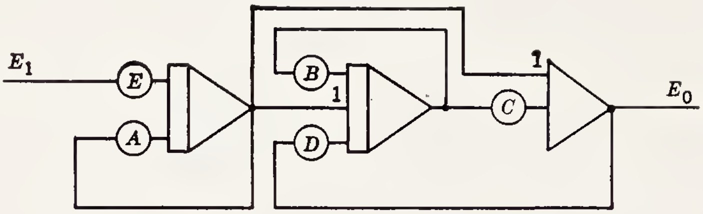
No.	Bode plot	Transfer function	Time constants	Gains
				
13		$\frac{K(T_2s+1)^2}{(T_1s+1)(T_3s+1)}$	$T_3 = \frac{1}{B}$ $T_2 = \frac{1}{B-C} = \frac{1}{D-F}$ $T_1 = \frac{1}{D-EF}$ $K = \frac{T_1T_3}{(T_2)^2}$	$A = 1$ $B = \frac{1}{T_3}$ $B - C = D - F = \frac{1}{T_2}$ $D - EF = \frac{1}{T_1}$
14		$\frac{K(T_1s+1)(T_4s+1)}{(T_2s+1)(T_3s+1)}$	$T_1 = \frac{1}{B-C}$ $T_2 = \frac{1}{B}$ $T_3 = \frac{1}{D-EF}$ $T_4 = \frac{1}{D-F}$ $K = \frac{T_2T_3}{T_1T_4}$	$A = 1$ $B = \frac{1}{T_2}$ $B - C = \frac{1}{T_1}$ $D - F = \frac{1}{T_4}$ $D - EF = \frac{1}{T_3}$
15		$\frac{(T_1s+1)(T_4s+1)}{(T_2s+1)(T_3s+1)}$	$T_1 = \frac{1}{B-C}$ $T_2 = \frac{1}{B}$ $T_3 = \frac{1}{D-EF}$ $T_4 = \frac{1}{D-F}$	$A = \frac{T_1T_4}{T_2T_3}$ $B = \frac{1}{T_2}$ $B - C = \frac{1}{T_1}$ $D - F = \frac{1}{T_4}$ $D - EF = \frac{1}{T_3}$

TABLE 3-1 (*Continued*)

<table>
<tr><th>No.</th><th>Bode plot</th><th>Transfer function</th><th>Time constants</th><th>Gains</th></tr>
<tr><td colspan="5">
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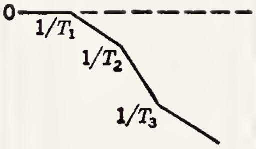
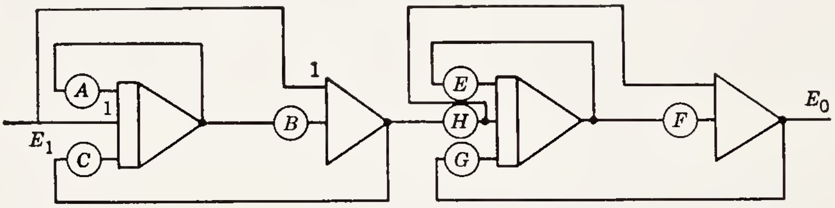
<tr><td>16</td><td>
</td><td>$\frac{T_3s + 1}{(T_1s + 1)(T_2s + 1)}$</td><td>$T_1 = \frac{1}{A}$
$T_2 = \frac{1}{B - CD}$
$T_3 = \frac{1}{B - C}$</td><td>$A = \frac{1}{T_1}$
$B = C + \frac{1}{T_3}$
$D = \frac{1}{CT_3} - \frac{1}{CT_2} + 1$
$E = \frac{T_3}{T_1T_2}$</td></tr>
<tr><td colspan="5">
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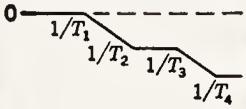
<tr><td>17</td><td>
</td><td>$\frac{(T_2s + 1)(T_4s + 1)}{(T_1s + 1)(T_3s + 1)}$</td><td>$T_1 = \frac{1}{A - BC}$
$T_2 = \frac{1}{A - B}$
$T_3 = \frac{1}{E - FG}$
$T_4 = \frac{1}{E - F}$</td><td>$A = B + \frac{1}{T_2}$
$C = \frac{T_1 - T_2}{BT_1T_2} + 1$
$E = F + \frac{1}{T_4}$
$G = \frac{T_3 - T_4}{FT_3T_4}$
$H = \frac{T_2T_4}{T_1T_3}$</td></tr>
</table>

TABLE 3-1 (*Continued*)

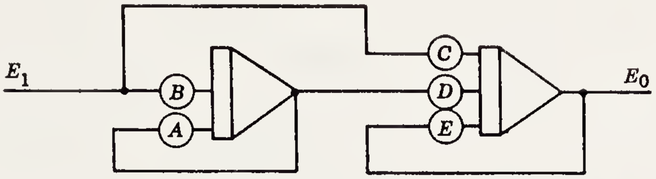
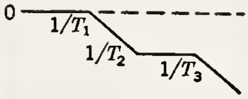
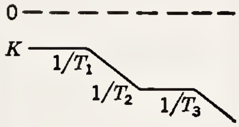
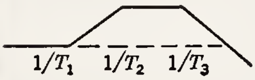
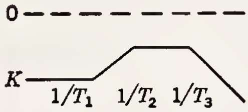
No.	Bode plot	Transfer function	Time constants	Gains
				
18		$\dfrac{T_2s+1}{(T_1s+1)(T_3s+1)}$	$T_1=\dfrac{1}{E}$ $T_2=\dfrac{E}{EA-D}$ $T_3=\dfrac{1}{A}$	$A=\dfrac{1}{T_3}$ $B=AT_2$ $C=BE$ $D=EA-\dfrac{E}{T_2}$ $E=\dfrac{1}{T_1}$
19		$\dfrac{K(T_2s+1)}{(T_1s+1)(T_3s+1)}$	$T_1=\dfrac{1}{E}$ $T_2=\dfrac{E}{EA-D}$ $T_3=\dfrac{1}{A}$ $K=\dfrac{T_3B}{T_2}$	$A=\dfrac{1}{T_3}$ $B=KAT_2$ $C=BE$ $D=EA-\dfrac{E}{T_2}$ $E=\dfrac{1}{T_1}$
20		$\dfrac{T_1s+1}{(T_2s+1)(T_3s+1)}$	$T_1=\dfrac{E}{EA-D}$ $T_2=\dfrac{1}{E}$ $T_3=\dfrac{1}{A}$	$A=\dfrac{1}{T_3}$ $B=AT_1$ $C=BE$ $D=EA-\dfrac{E}{T_1}$ $E=\dfrac{1}{T_2}$
21		$\dfrac{K(T_1s+1)}{(T_2s+1)(T_3s+1)}$	$T_1=\dfrac{E}{EA-D}$ $T_2=\dfrac{1}{E}$ $T_3=\dfrac{1}{A}$ $K=\dfrac{T_3B}{T_1}$	$A=\dfrac{1}{T_3}$ $B=KAT_1$ $C=BE$ $D=EA-\dfrac{E}{T_1}$ $E=\dfrac{1}{T_2}$

TABLE 3-1 (*Continued*)

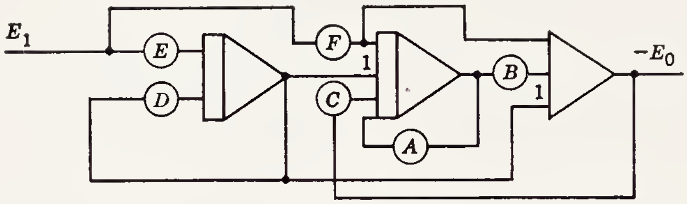
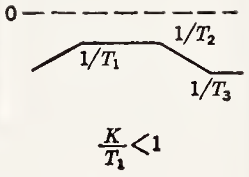
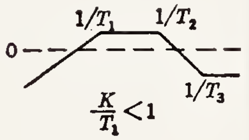
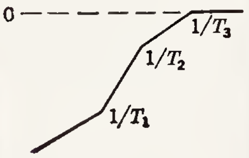
No.	Bode plot	Transfer function	Time constants	Gains
				
22		$\dfrac{Ks(T_3s+1)}{(T_1s+1)(T_2s+1)}$	$T_1 = \dfrac{1}{D}$ $T_2 = \dfrac{1}{A-BC}$ $T_3 = \dfrac{1}{A-B}$ $K = \dfrac{T_2FT_1}{T_3}$	$A = B + \dfrac{1}{T_3}$ $C = \dfrac{T_2 - T_3}{BT_2T_3} + 1$ $D = \dfrac{1}{T_1}$ $E = DF$ $F = \dfrac{T_3K}{T_2T_1}$
23		$\dfrac{Ks(T_3s+1)}{(1+T_1s)(1+T_2s)}$	$T_1 = \dfrac{1}{D}$ $T_2 = \dfrac{1}{A-BC}$ $T_3 = \dfrac{1}{A-B}$ $K = \dfrac{T_2T_1F}{T_3}$	$A = B + \dfrac{1}{T_3}$ $C = \dfrac{T_2 - T_3}{BT_2T_3} + 1$ $D = \dfrac{1}{T_1}$ $E = DF$ $F = \dfrac{T_3K}{T_2T_1}$
24		$\dfrac{Ks(T_1s+1)}{(T_2s+1)(T_3s+1)}$	$T_1 = \dfrac{1}{A-B}$ $T_2 = \dfrac{1}{A}$ $T_3 = \dfrac{1}{D}$	$A = \dfrac{1}{T_2}$ $B = \dfrac{T_2 - T_1}{T_1T_2}$ $D = \dfrac{1}{T_3}$ $E = F = \dfrac{T_1K}{T_2T_3} = 1$

TABLE 3-1 (*Continued*)

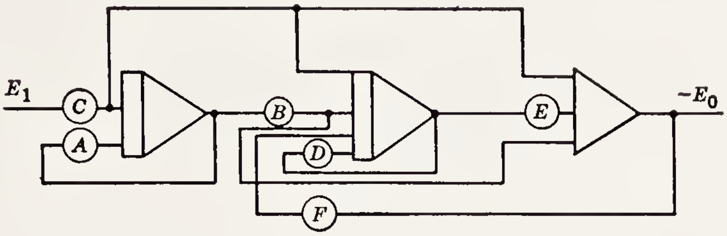
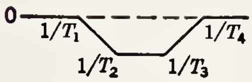
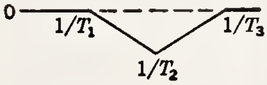
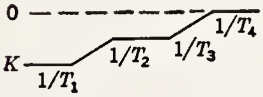
No.	Bode plot	Transfer function	Time constants	Gains
				
25		$\dfrac{(T_2s+1)(T_3s+1)}{(T_1s+1)(T_4s+1)}$	$T_1 = \dfrac{1}{D-EF}$ $T_2 = \dfrac{1}{D-E}$ $T_3 = \dfrac{1}{A-B}$ $T_4 = \dfrac{1}{A}$	$A = \dfrac{1}{T_4}$ $B = \dfrac{T_3-T_4}{T_3T_4}$ $C = \dfrac{T_2T_3}{T_1T_4} = 1$ $D = E + \dfrac{1}{T_2}$ $F = \dfrac{T_1-T_2}{ET_1T_2} + 1$
26		$\dfrac{(T_2s+1)^2}{(T_1s+1)(T_3s+1)}$	$T_1 = \dfrac{1}{D-EF}$ $T_2 = \dfrac{1}{D-E}$ $T_2 = \dfrac{1}{A-B}$ $T_3 = \dfrac{1}{A}$	$A = \dfrac{1}{T_3}$ $B = \dfrac{T_2-T_3}{T_2T_3}$ $C = 1$ $D = E + \dfrac{1}{T_2}$ $F = \dfrac{T_1-T_2}{ET_1T_2} + 1$
27		$\dfrac{K(T_1s+1)(T_3s+1)}{(T_2s+1)(T_4s+1)}$	$T_1 = \dfrac{1}{D-E}$ $T_2 = \dfrac{1}{D}$ $T_3 = \dfrac{1}{A-B}$ $T_4 = \dfrac{1}{A}$ $K = \dfrac{T_2T_4}{T_1T_3}$	$A = \dfrac{1}{T_4}$ $B = \dfrac{T_3-T_4}{T_3T_4}$ $C = 1$ $D = \dfrac{1}{T_2}$ $E = \dfrac{T_1-T_2}{T_1T_2}$ $F = 0$

Table 3-1 *(Continued)*

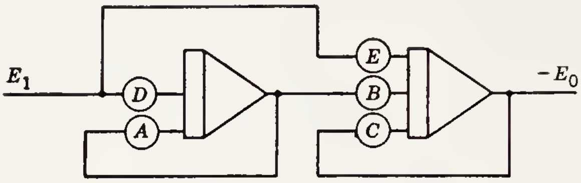
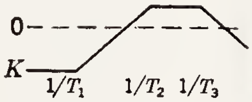
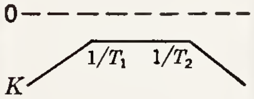
<table>
<tr><th>No.</th><th>Bode plot</th><th>Transfer function</th><th>Time constants</th><th>Gains</th></tr>
<tr><td colspan="5">
</td></tr>
<tr><td>28</td><td>
</td><td>$\frac{K(T_1s + 1)}{(T_2s + 1)(T_3s + 1)}$</td><td>$T_1 = \frac{1}{A - B}$
$T_2 = \frac{1}{A}$
$T_3 = \frac{1}{C}$
$K = \frac{DT_2T_3}{T_1}$</td><td>$A = \frac{1}{T_2}$
$B = \frac{T_1 - T_2}{T_1T_2}$
$C = \frac{1}{T_3}$
$D = E = \frac{KT_1}{T_2T_3}$</td></tr>
<tr><td>29</td><td>
</td><td>$\frac{Ks}{(T_1s + 1)(T_2s + 1)}$</td><td>$T_1 = \frac{1}{A}$
$= \frac{K}{D}$
$T_2 = \frac{1}{B}$
$= \frac{1}{C}$
$K = ET_2T_1$</td><td>$A = \frac{1}{T_1}$
$B = C = \frac{1}{T_2}$
$D = \frac{K}{T_1^2}$
$E = \frac{K}{T_2T_1}$</td></tr>
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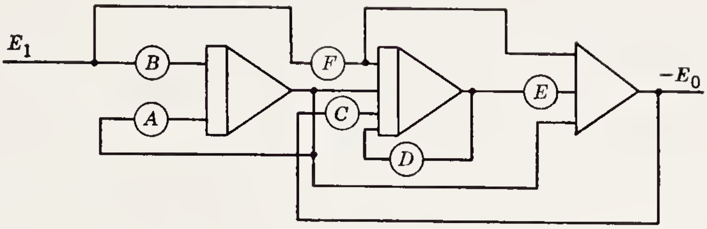
TABLE 3-1 (*Continued*)



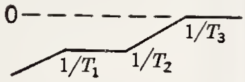
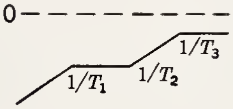
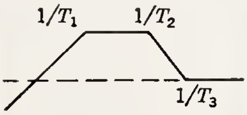
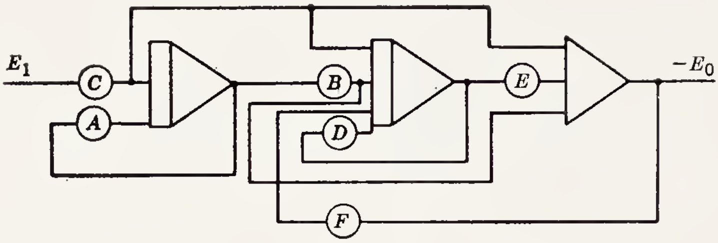
No.	Bode plot	Transfer function	Time constants	Gains
30		$\frac{Ks(T_2s + 1)}{(T_1s + 1)(T_3s + 1)}$	$T_1 = \frac{1}{A}$ $T_2 = \frac{1}{D - E}$ $T_3 = \frac{1}{D}$ $K = \frac{T_3T_1}{T_2}$	$A = \frac{1}{T_1}$ $B = AF$ $C = 0$ $D = \frac{1}{T_3}$ $E = \frac{T_2 - T_3}{T_2T_3}$
31		$\frac{Ks(T_2s + 1)}{(T_1s + 1)(T_3s + 1)}$	$T_1 = \frac{1}{A}$ $T_2 = \frac{1}{D - E}$ $T_3 = \frac{1}{D}$ $K = \frac{T_3FT_1}{T_2}$	$A = \frac{1}{T_1}$ $B = AF$ $C = 0$ $D = \frac{1}{T_3}$ $E = \frac{T_2 - T_3}{T_2T_3}$ $F = \frac{T_2K}{T_3T_1}$
32		$\frac{Ks(T_3s + 1)}{(T_1s + 1)(T_2s + 1)}$	$T_1 = \frac{1}{A}$ $T_2 = \frac{1}{D - EC}$ $T_3 = \frac{1}{D - E}$ $K = \frac{T_2FT}{T_3}$	$A = \frac{1}{T_1}$ $B = AF$ $C = \frac{T_2T_3}{ET_2T_3}$ $D = E + \frac{1}{T_3}$ $F = \frac{T_3K}{T_1T_2} = 1$

TABLE 3-1 (*Continued*)



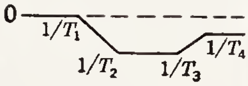
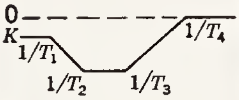
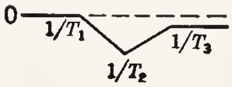
No.	Bode plot	Transfer function	Time constants	Gains
33		$\frac{(T_2s + 1)(T_3s + 1)}{(T_1s + 1)(T_4s + 1)}$	$T_1 = \frac{1}{D - EF}$ $T_2 = \frac{1}{D - E}$ $T_3 = \frac{1}{A - B}$ $T_4 = \frac{1}{A}$	$A = \frac{1}{T_4}$ $B = \frac{T_3 - T_4}{T_3T_4}$ $C = \frac{T_2T_3}{T_1T_4}$ $D = E + \frac{1}{T_2}$ $F = \frac{T_1 - T_2}{ET_1T_2} + 1$
34		$\frac{K(T_2s + 1)(T_3s + 1)}{(T_1s + 1)(T_4s + 1)}$	$T_1 = \frac{1}{D - EF}$ $T_2 = \frac{1}{D - E}$ $T_3 = \frac{1}{A - B}$ $T_4 = \frac{1}{A}$ $K = \frac{T_1T_4}{T_2T_3}$	$A = \frac{1}{T_4}$ $B = \frac{T_3 - T_4}{T_3T_4}$ $C = 1$ $D = E + \frac{1}{T_2}$ $F = \frac{T_1 - T_2}{ET_1T_2} + 1$
35		$\frac{(T_2s + 1)^2}{(T_1s + 1)(T_3s + 1)}$	$T_1 = \frac{1}{D - EF}$ $T_2 = \frac{1}{D - E}$ $= \frac{1}{A - B}$ $T_3 = \frac{1}{A}$ $(T_2)^2 < T_2T_3$	$A = \frac{1}{T_3}$ $B = \frac{T_2 - T_3}{T_2T_3}$ $C = \frac{(T_2)^2}{T_1T_3}$ $D = E + \frac{1}{T_2}$ $F = \frac{T_1 - T_2}{ET_1T_2} + 1$

TABLE 3-1 (*Continued*)

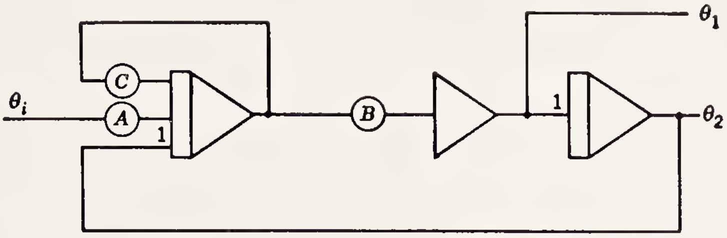
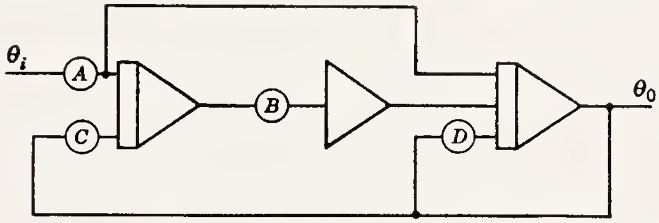
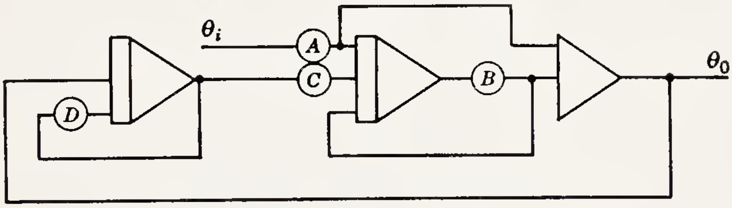
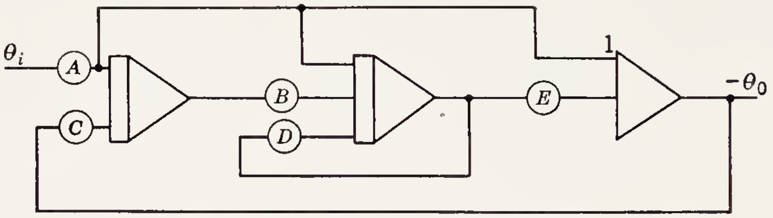
No.	Transfer function	Time constants	Gains
36			
	$\frac{\theta_1}{\theta_i} = \frac{Ks}{T^2s^2 + 2hTs + 1}$ $\frac{\theta_2}{\theta_i} = \frac{-K}{T^2s^2 + 2hTs + 1}$	$K = A$ $T = \frac{1}{\sqrt{B}}$ $h = \frac{C}{2\sqrt{B}}$	$A = K$ $B = \frac{1}{T^2}$ $C = \frac{2h}{T}$
37			
	$\frac{\theta_0}{\theta_i} = \frac{-K(T_1s + 1)}{T_2^2s^2 + 2hT_2s + 1}$	$K = \frac{A}{C}$ $T_1 = \frac{1}{B}$ $T_2 = \frac{1}{\sqrt{BC}}$ $h = \frac{DT_2}{2}$	$A = CK$ $B = \frac{1}{T_1}$ $C = \frac{T_1}{T_2^2}$ $D = \frac{2h}{T_2}$
38			
	$\frac{\theta_0}{\theta_i} = \frac{Ks(T_1s + 1)}{T_2^2s^2 + 2hT_2s + 1}$	$K = \frac{AD}{B(D + C)}$ $T_1 = \frac{1}{D}$ $T_2 = \frac{1}{\sqrt{B(D + C)}}$ $h = \frac{B + D}{2\sqrt{B(D + C)}}$	$A = \frac{KB(D + C)}{D}$ $B = \frac{2h}{T_2} - \frac{1}{T_1}$ $C = \frac{T_2^2 + T_1^2 - 2hT_1T_2}{T_1T_2(2hT_1 - T_2)}$ $D = \frac{1}{T_1}$

TABLE 3-1 (*Continued*)

No.	Transfer function	Time constants	Gains
			
39	$\dfrac{\theta_0}{\theta_i} = \dfrac{-K(T_1{}^2s^2 + 2h_1T_1s + 1)}{T_2{}^2s^2 + 2h_2T_2s + 1}$	$K = \dfrac{A}{C}$ $T_1 = \dfrac{1}{\sqrt{BE}}$ $h_1 = \dfrac{D - E}{2\sqrt{BE}}$ $T_2 = \dfrac{1}{\sqrt{BCE}}$ $h_2 = \dfrac{D}{2\sqrt{BCE}}$	$A = CK$ $B = \dfrac{T_2}{2T_1(h_2T_1 - h_1T_2)}$ $C = \dfrac{T_1{}^2}{T_2{}^2}$ $D = \dfrac{2h_2}{T_2}$ $E = 2\left(\dfrac{h_2}{T_2} - \dfrac{h_1}{T_1}\right)$

for the circuit have the values

$$\begin{aligned} T_1 &= 1.00 \text{ sec} \\ T_2 &= 0.50 \text{ sec} \\ T_3 &= 0.20 \text{ sec} \\ K &= \tfrac{1}{2} \end{aligned}$$

Substituting these values into the gain equations gives

$$\begin{aligned} A &= \frac{1}{0.2} = 5.0 \\ B &= \tfrac{5}{2} = 2.5 \\ C &= 2.5 \times 2 = 5.0 \\ D &= 2 \times 5 - 2 = 8.0 \\ E &= \frac{1}{0.5} = 2.0 \end{aligned}$$

The circuit which results is shown in Fig. 3-8. It can be seen that it was possible to omit the potentiometers A and C since the gains required corresponded exactly to those available at the amplifier input. It can be seen from this example that the gains indicated in the table can be

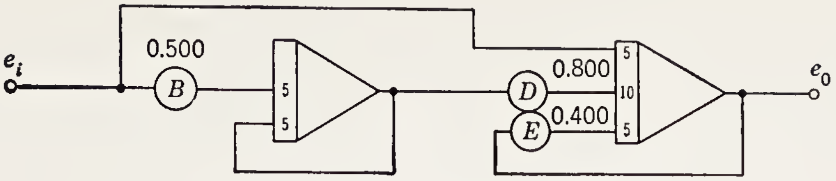


FIG. 3-8. A circuit for the simulation of

$$\frac{E_o}{E_i} = \frac{\frac{1}{2}(1 + s)}{(1 + 0.5s)(1 + 0.2s)}$$

considered as a potentiometer setting times an input gain. The method shown here is subject to limitations of available gain. To illustrate this, consider the use of the same circuit (21) to simulate the transfer function

$$\frac{E_o}{E_i} = \frac{\frac{1}{2}(1 + 0.2s)}{(1 + 0.1s)(1 + 0.05s)} \tag{3-18}$$

In this case, the gains required become

$$\begin{aligned} A &= \frac{1}{0.05} = 20 \\ B &= \tfrac{1}{2} \times 20 \times 0.2 = 2 \\ C &= 2 \times 10 = 20 \\ D &= 10 \times 20 - 10 \times 5 = 150 \\ E &= \frac{1}{0.10} = 10 \end{aligned}$$

Since a gain of 150 (for D) cannot be achieved on an amplifier of the fixed-gain type, this circuit cannot be used to simulate Eq. (3-18).

3-5. ANALYSIS OF A FIVE-IMPEDANCE NETWORK

In analog-computer work the question often arises as to just what the transfer function of a particular circuit is. In general, this question can be answered easily by applying the techniques of nodal analysis to the circuit. These techniques were used in Chap. 1 to show how a d-c amplifier can be used to sum and to integrate. We shall now show how to apply these techniques to analyze more complicated circuits. As an example, consider the circuit of Fig. 3-9, which contains one amplifier and five two-terminal impedances. For the time being, we shall assume these impedances to be completely general. Then, if we note that point 2 is the grid of the amplifier and is therefore virtually at ground potential,* and if we assume that no current flows through the amplifier, we can

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* See Sec. 8-2.

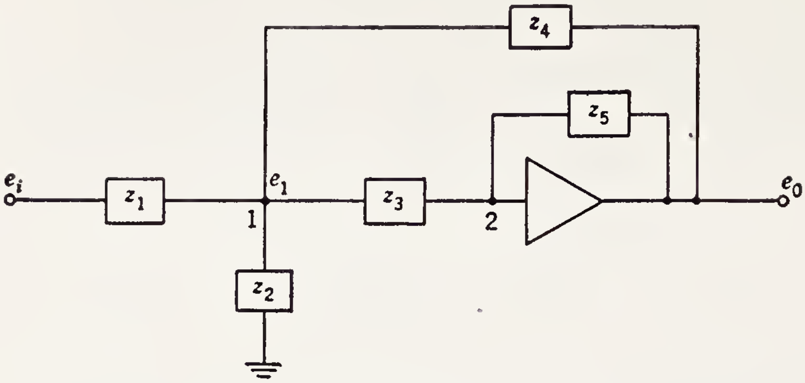


FIG. 3-9. A five-impedance network.

write for nodal equations at point 1 and point 2, respectively,

$$\frac{e_1 - e_i}{Z_1} + \frac{e_1}{Z_2} + \frac{e_1}{Z_3} + \frac{e_1 - e_o}{Z_4} = 0 \tag{3-19}$$

and

$$\frac{e_1}{Z_3} + \frac{e_o}{Z_5} = 0 \tag{3-20}$$

Combining these equations and eliminating e_1, we can write the transfer function of this circuit as

$$\frac{e_o}{e_i} = \frac{-Z_4/Z_1}{1 + \frac{Z_3Z_4}{Z_5}\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \frac{1}{Z_4}\right)} \tag{3-21}$$

This transfer function holds no matter what the transfer functions of the individual impedances. As an example, let us consider the particular case where Z_1, Z_3, and Z_5 are resistances and Z_2 and Z_4 are capacitances. We shall assign values to be

$$Z_1 = R_1 \qquad Z_2 = \frac{1}{C_1 s} \qquad Z_3 = R_2$$

$$Z_4 = \frac{1}{C_2 s} \qquad Z_5 = R_3$$

Then from Eq. (3-21)

$$\begin{aligned}\frac{E_o}{E_i} &= \frac{-1/R_1C_2s}{1 + \frac{R_2}{R_3C_2s}\left(\frac{1}{R_1} + C_1s + \frac{1}{R_2} + C_2s\right)} \\ &= -\frac{R_3/(R_1 + R_2)}{1 + \frac{R_1}{R_1 + R_2}[R_2C_1 + C_2(R_2 + R_3)]s}\end{aligned} \tag{3-22}$$

Thus this circuit, which is shown as Fig. 3-10, can be used to simulate a first-order lag response. This circuit is presented only as an illustration of the method; it will be shown in Sec. 3-6 that this is not the optimum circuit for this task. However, if we let Z_1, Z_3, and Z_4 be resistances and Z_2 and Z_5 capacitances, we have a circuit of wide utility. For instance, consider the circuit of Fig. 3-11, where we have let

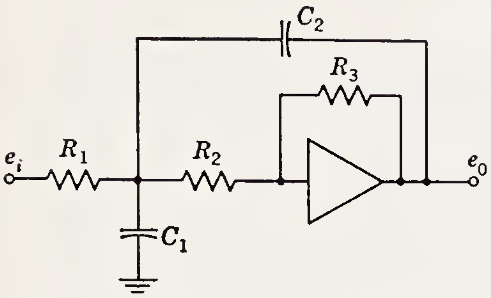


FIG. 3-10. A first-order lag circuit.

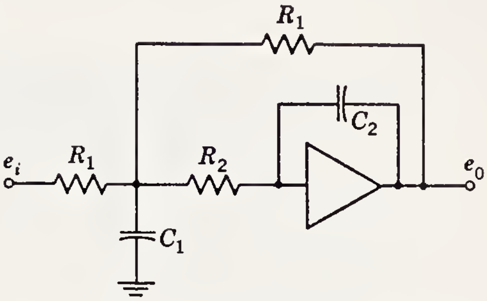


FIG. 3-11. A circuit for simulating a second-order response.

$$Z_1 = Z_4 = R_1 \qquad Z_3 = R_2$$
$$Z_2 = \frac{1}{C_1 s} \qquad Z_5 = \frac{1}{C_2 s}$$

From Eq. (3-21) we can write

$$\frac{E_o}{E_i} = \frac{-1}{1 + (R_1 + 2R_2)C_2 s + R_1 R_2 C_1 C_2 s^2} \tag{3-23}$$

It is evident that Eq. (3-23) is of the form

$$\frac{E_o}{E_i} = \frac{-1}{1 + (2\xi/\omega_n)s + s^2/\omega_n{}^2}$$

We can write the circuit impedance values in terms of our response parameters as

$$\frac{1}{\omega_n{}^2} = R_1 R_2 C_1 C_2$$

and

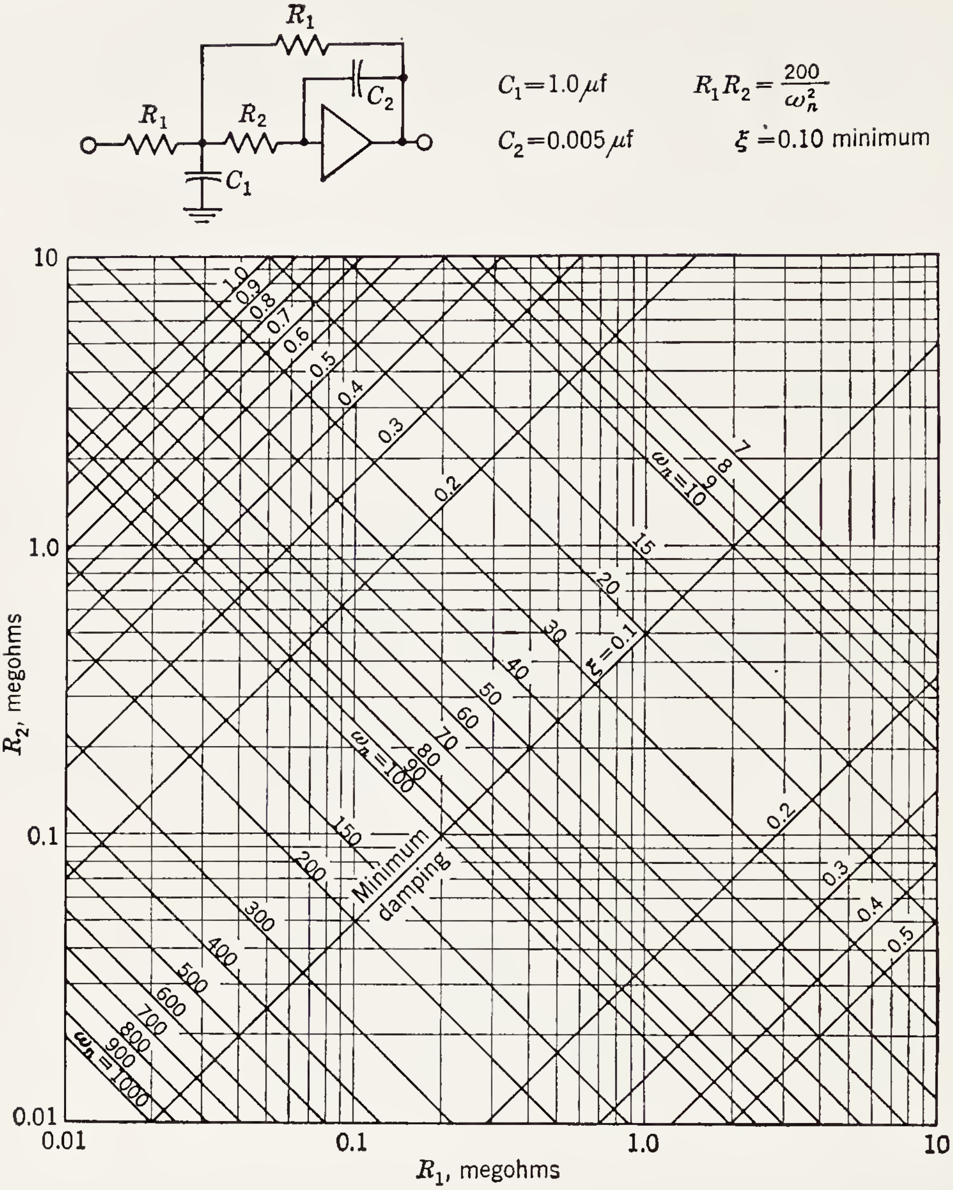
$$\frac{2\xi}{\omega_n} = (R_1 + 2R_2)C_2$$

Combining and rewriting, we get

$$\xi = \sqrt{\frac{(R_1 + 2R_2)^2}{4R_1 R_2} \frac{C_2}{C_1}} \tag{3-24}$$

and

$$\omega_n = \sqrt{\frac{1}{R_1 R_2 C_1 C_2}} \tag{3-25}$$

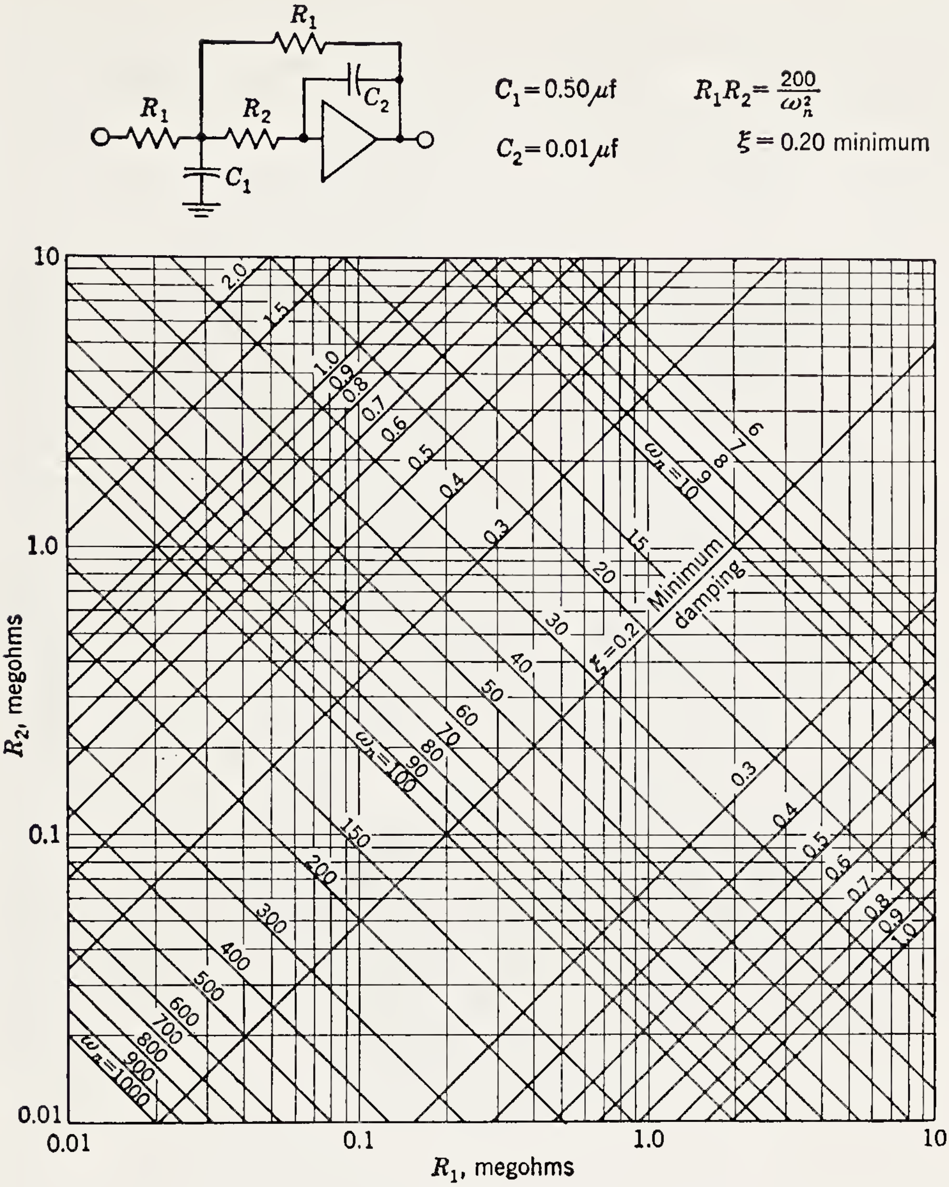


(a)

Fig. 3-12. Nomograph for second-order system.

We can now establish limits on the values of the parameters to be simulated. From Eq. (3-25) it can be seen that the magnitude of ω_n will be limited only by the consideration of available components of very large or very small size. The same argument holds for the maximum value of ξ. The establishment of a lower limit on ξ, however, requires some further calculation. To this end, let $R_2 = kR_1$, and Eq. (3-24) becomes

$$\xi = \frac{1 + 2k}{\sqrt{k}} \sqrt{\frac{C_2}{4C_1}} \qquad (3\text{-}26)$$



(*b*)

FIG. 3-12 (*Continued*)

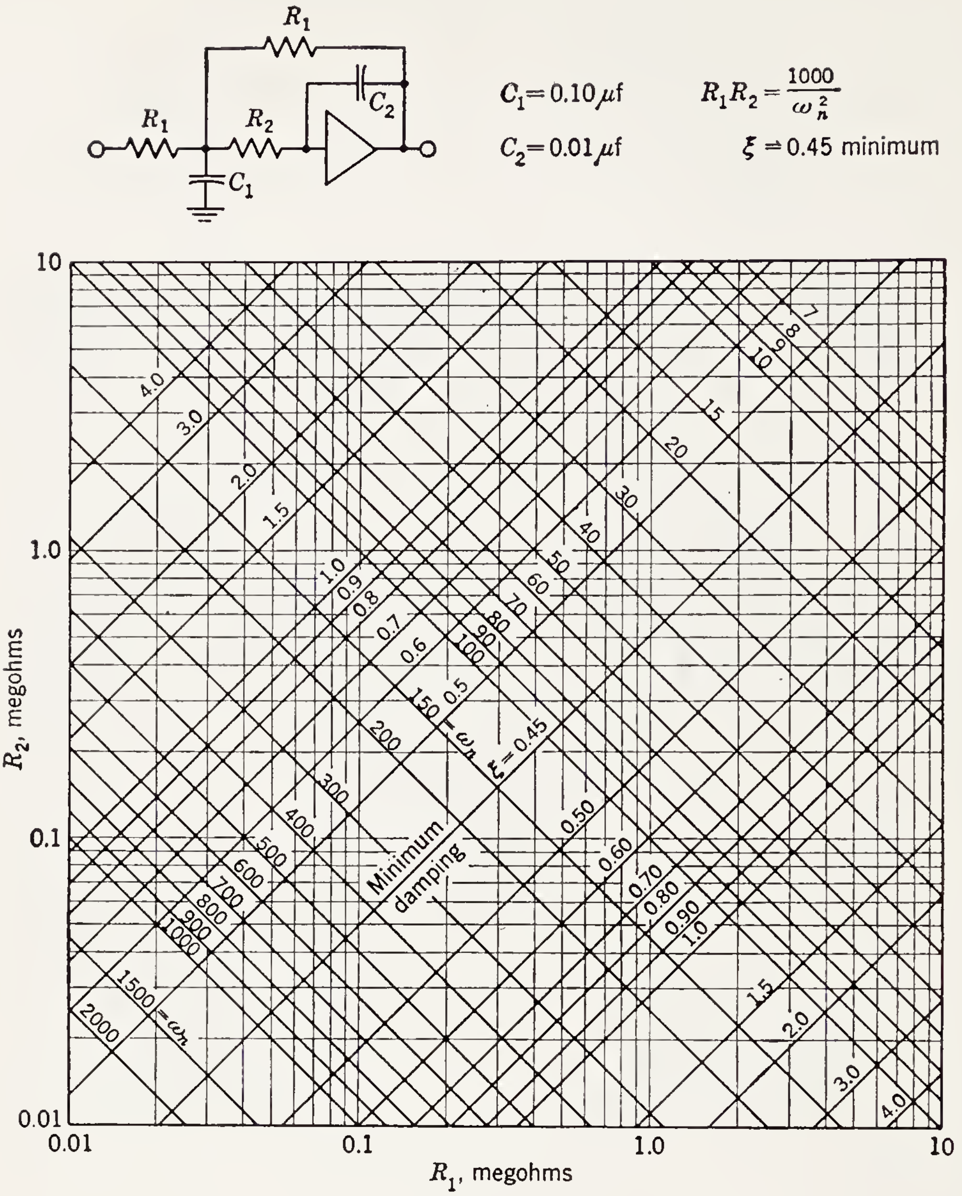
If we take the derivative of ξ with respect to k and equate it to zero, we have

$$\frac{d\xi}{dk} = \frac{k - \frac{1}{2}}{k^{3/2}} \sqrt{\frac{C_2}{4C_1}} = 0 \qquad (3\text{-}27)$$

Since Eq. (3-27) has the root $k = \frac{1}{2}$, we have

$$\xi_{\min} = \sqrt{\frac{2C_2}{C_1}} \qquad (3\text{-}28)$$

Thus the minimum damping ratio of this circuit is proportional to the ratio of the capacitances.



(c)

Fig. 3-12 (*Continued*)

This circuit is extremely helpful to the computer operator for two reasons: first, it produces a considerable saving in equipment, and, second, it eliminates the gain problems which accompany the solution of high-frequency systems by other methods. The disadvantages of the circuit are a decreased flexibility as far as parameter changes are concerned, a requirement for unusual component values, and a somewhat greater computational complexity. The latter problem can be alleviated by the preparation of nomographs relating circuit components to desired system parameters. Since decade resistors are cheaper and more common than decade capacitances, it is most convenient to prepare these nomographs for fixed capacitance values. A set of nomographs for this circuit is shown in Fig. 3-12*a* to *c*.* The resistance values R_1 and R_2 are determined by the intersection of the damping curve ξ and the frequency curve ω_n. It will be noticed that for any given set of values for ξ and ω_n there are two sets of resistance values. The use of these nomographs will be illustrated by the following example.

Example. A rate gyro has a natural frequency of 8 cps and a damping ratio of 0.7. That is, its dynamic characteristics can be approximated by the transfer function

$$\frac{E_o}{E_i} = \frac{1}{1 + 0.0279s + 0.000396s^2} \tag{3-29}$$

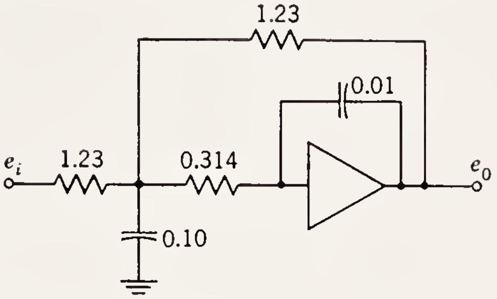


FIG. 3-13. Simulation of a second-order rate gyro.

Since the damping ratio is greater than 0.45, we use Fig. 3-12*c* to determine our circuit parameters. Converting the natural frequency to radians per second, we get $\omega_n = 50.1$. Reading from the intersections of the two lines for $\xi = 0.7$ and the line for $\omega_n = 50$, we find two possible sets of values for R_1 and R_2. These are

$R_1 = 0.314$ megohm $\quad R_2 = 1.23$ megohms

and $\quad R_1 = 2.45$ megohms $\quad R_2 = 0.161$ megohm

Both these sets are of a practical magnitude; either could be used satisfactorily. The circuit resulting from the use of the first set is shown as Fig. 3-13.

3-6. NETWORK SYNTHESIS USING AN IMPEDANCE TABLE

As has been illustrated in many texts, it is possible to obtain, by synthesis, networks to simulate transfer functions satisfying certain restrictions. We shall not discuss the general techniques of network synthesis but shall consider the special case where the transfer functions of certain component networks are given. The synthesis procedure is then to combine these networks in such a manner that the desired transfer function is obtained for the entire circuit.

The characteristics of a network are expressed by its short-circuit transfer impedance function. The short-circuit transfer impedance function Z of a network is defined as the input voltage divided by the output current with the output short-circuited. It can be seen from this definition that we can assign to every network a particular transfer imped-

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* The authors are indebted to H. J. Shear of the Ryan Aeronautical Co. for the preparation of these figures.

TABLE 3-2

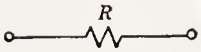
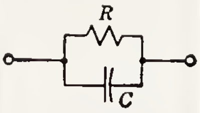
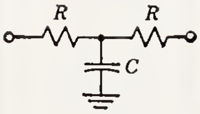
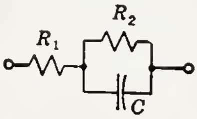
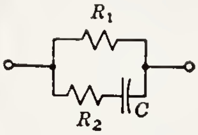
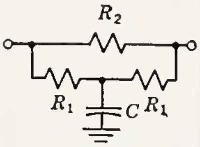
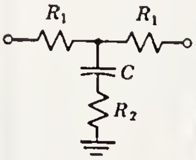
Transfer impedance function	Network	Relations	Inverse relations
A	1 	$A = R$	$R = A$
$\dfrac{A}{1 + sT}$	2 	$A = R$ $T = RC$	$R = A$ $C = \dfrac{T}{A}$
$A(1 + sT)$	3 	$A = 2R$ $T = \dfrac{RC}{2}$	$R = \dfrac{A}{2}$ $C = \dfrac{4T}{A}$
$A\,\dfrac{1 + s\theta T}{1 + sT}$ $\theta < 1$	4 	$A = R_1 + R_2$ $T = R_2C$ $\theta = \dfrac{R_1}{R_1 + R_2}$	$R_1 = A\theta$ $R_2 = A(1 - \theta)$ $C = \dfrac{T}{A(1 - \theta)}$
	5 	$A = R_1$ $T = (R_1 + R_2)C$ $\theta = \dfrac{R_2}{R_1 + R_2}$	$R_1 = A$ $R_2 = \dfrac{A\theta}{1 - \theta}$ $C = \dfrac{T(1 - \theta)}{A}$
$A\,\dfrac{1 + sT}{1 + s\theta T}$ $\theta < 1$	6 	$A = \dfrac{2R_1R_2}{2R_1 + R_2}$ $T = \dfrac{R_1C}{2}$ $\theta = \dfrac{2R_1}{2R_1 + R_2}$	$R_1 = \dfrac{A}{2(1 - \theta)}$ $R_2 = \dfrac{A}{\theta}$ $C = \dfrac{4T(1 - \theta)}{A}$
	7 	$A = 2R_1$ $T = \left(R_2 + \dfrac{R_1}{2}\right)C$ $\theta = \dfrac{2R_2}{2R_2 + R_1}$	$R_1 = \dfrac{A}{2}$ $R_2 = \dfrac{A\theta}{4(1 - \theta)}$ $C = \dfrac{4T(1 - \theta)}{A}$

TABLE 3-2 (*Continued*)

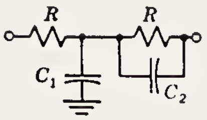
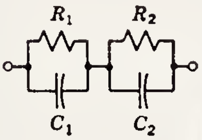
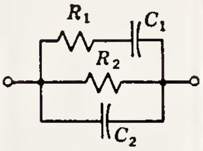
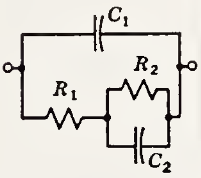
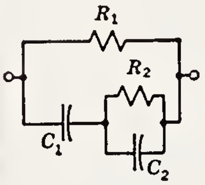
Transfer impedance function	Network	Relations	Inverse relations
$A\dfrac{1+sT}{1+s\theta T}$ $\theta < 1$	8 	$A = 2R$ $T = \dfrac{R}{2}(C_1 + C_2)$ $\theta = \dfrac{2C_2}{C_1 + C_2}$	$R = \dfrac{A}{2}$ $C_1 = \dfrac{2T(2-\theta)}{A}$ $C_2 = \dfrac{2T\theta}{A}$
$A\dfrac{1+sT_2}{(1+sT_1)(1+sT_3)}$ $T_1 < T_2 < T_3$	9 	$A = R_1 + R_2$ $T_1 = R_1C_1$ $T_2 = \dfrac{R_1R_2}{R_1 + R_2}(C_1 + C_2)$ $T_3 = R_2C_2$	$R_1 = \dfrac{A(T_2 - T_1)}{T_3 - T_1}$ $R_2 = \dfrac{A(T_3 - T_2)}{T_3 - T_1}$ $C_1 = \dfrac{T_1(T_3 - T_1)}{A(T_2 - T_1)}$ $C_2 = \dfrac{T_3(T_3 - T_1)}{A(T_3 - T_2)}$
	10 	$A = R_2$ $T_2 = R_1C_1$ $T_1T_3 = R_1R_2C_1C_2$ $T_1 + T_3 = R_1C_1 + R_2C_2 + R_2C_1$	$R_1 = \dfrac{AT_2^2}{(T_3 - T_2)(T_2 - T_1)}$ $R_2 = A$ $C_1 = \dfrac{(T_3 - T_2)(T_2 - T_1)}{AT_2}$ $C_2 = \dfrac{T_1T_3}{AT_2}$
	11 	$A = R_1 + R_2$ $T_2 = \dfrac{R_1R_2}{R_1 + R_2}C_2$ $T_1T_3 = R_1R_2C_1C_2$ $T_1 + T_3 = R_1C_1 + R_2C_2 + R_2C_1$	$R_1 = \dfrac{AT_2^2}{T_1T_2 + T_2T_3 - T_1T_3}$ $R_2 = \dfrac{A(T_3 - T_2)(T_2 - T_1)}{T_1T_2 + T_2T_3 - T_1T_3}$ $C_1 = \dfrac{T_1T_3}{AT_2}$ $C_2 = \dfrac{(T_1T_2 + T_2T_3 - T_1T_3)^2}{AT_2(T_3 - T_2)(T_2 - T_1)}$
	12 	$A = R_1$ $T_2 = R_2(C_1 + C_2)$ $T_1T_3 = R_1R_2C_1C_2$ $T_1 + T_3 = R_1C_1 + R_2C_2 + R_2C_1$	$R_1 = A$ $R_2 = \dfrac{A(T_3 - T_2)(T_2 - T_1)}{(T_1 + T_3 - T_2)^2}$ $C_1 = \dfrac{T_1 + T_3 - T_2}{A}$ $C_2 = \dfrac{T_1T_3(T_1 + T_3 - T_2)}{A(T_3 - T_2)(T_2 - T_1)}$

TABLE 3-2 (*Continued*)

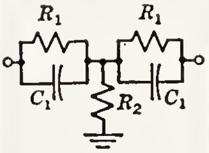
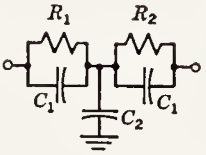
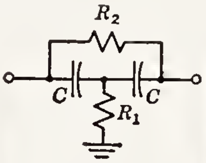
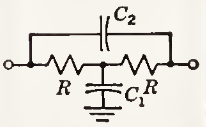
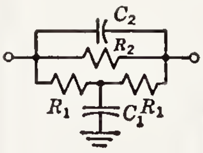
Transfer impedance function	Network	Relations	Inverse relations
$A\dfrac{1+sT_2}{(1+sT_1)(1+sT_3)}$ $T_2 \leq T_1 \leq T_3$	13 	$A \doteq 2R_1 + \dfrac{R_1^2}{R_2}$ $T_1 = R_1C_1$ $T_2 = \dfrac{R_1R_2}{R_1+2R_2}(C_1+C_2)$ $T_3 = R_1C_2$	$R_1 = \dfrac{AT_2}{T_1+T_3}$ $R_2 = \dfrac{AT_2^2}{(T_1+T_3)(T_1+T_3-2T_2)}$ $C_1 = \dfrac{T_1(T_1+T_3)}{AT_2}$ $C_2 = \dfrac{T_3(T_1+T_3)}{AT_2}$
$A\dfrac{1+sT_2}{(1+sT_1)(1+sT_3)}$ $T_1 \leq T_3 \leq T_2$	14 	$A = R_1 + R_2$ $T_1 = R_1C_1$ $T_2 = \dfrac{R_1R_2}{R_1+R_2}(2C_1+C_2)$ $T_3 = R_2C_1$	$R_1 = \dfrac{AT_1}{T_1+T_3}$ $R_2 = \dfrac{AT_3}{T_1+T_3}$ $C_1 = \dfrac{T_1+T_3}{A}$ $C_2 = \dfrac{T_1+T_3}{A}\left(\dfrac{T_2}{T_3}+\dfrac{T_2}{T_1}-2\right)$
$A\dfrac{1+sT_1}{1+sT_1+s^2T_1T_2}$	15 	$A = R_2$ $T_1 = 2R_1C$ $T_2 = \dfrac{R_2C}{2}$	$R_1 = \dfrac{AT_1}{4T_2}$ $R_2 = A$ $C = \dfrac{2T_2}{A}$
$A\dfrac{1+sT_2}{1+sT_1+s^2T_1T_2}$	16 	$A = 2R$ $T_1 = 2RC_2$ $T_2 = \dfrac{RC_1}{2}$	$R = \dfrac{A}{2}$ $C_1 = \dfrac{4T_2}{A}$ $C_2 = \dfrac{T_1}{A}$
$A\dfrac{1+sT_3}{1+sT_1+s^2T_1T_2}$ $T_2 > \dfrac{T_1}{4}\left(\begin{matrix}\text{complex}\\ \text{roots}\end{matrix}\right)$ $T_3 > T_2$	17 	$A = \dfrac{2R_1R_2}{2R_1+R_2}$ $T_1 = \dfrac{R_1(R_1C_1+2R_2C_2)}{2R_1+R_2}$ $T_2 = \dfrac{R_1R_2C_1C_2}{R_1C_1+2R_2C_2}$ $T_3 = \dfrac{R_1C_1}{2}$	$R_1 = \dfrac{AT_3^2}{2[T_3^2 - T_1(T_3-T_2)]}$ $R_2 = \dfrac{AT_3^2}{T_1(T_3-T_2)}$ $C_1 = \dfrac{4[T_3^2 - T_1(T_3-T_2)]}{AT_3}$ $C_2 = \dfrac{T_1T_2}{AT_3}$

TABLE 3-2 (*Continued*)

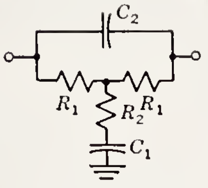
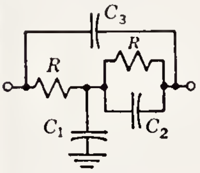
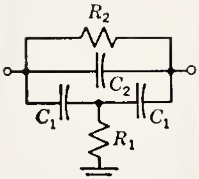
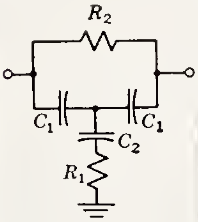
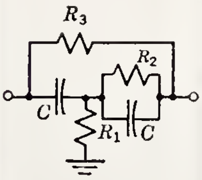
Transfer impedance function	Network	Relations	Inverse relations
$A\,\dfrac{1+sT_3}{1+sT_1+s^2T_1T_2}$ $T_2 > \dfrac{T_1}{4}\left(\text{complex roots}\right)$ $T_3 > T_2$	18 	$A = 2R_1$ $T_1 = R_2C_1 + 2R_1C_2$ $T_2 = \dfrac{R_1(R_1+2R_2)C_1C_2}{R_2C_1+2R_1C_2}$ $T_3 = \left(R_2 + \dfrac{R_1}{2}\right)C_1$	$R_1 = \dfrac{A}{2}$ $R_2 = \dfrac{AT_1(T_3-T_2)}{4[T_3^2 - T_1(T_3-T_2)]}$ $C_1 = \dfrac{4[T_3^2 - T_1(T_3-T_2)]}{AT_3}$ $C_2 = \dfrac{T_1T_2}{AT_3}$
	19 	$A = 2R$ $T_1 = R(C_2 + 2C_3)$ $T_2 = \dfrac{RC_3(C_1+C_2)}{C_2+2C_3}$ $T_3 = \dfrac{R}{2}(C_1+C_2)$	$R = \dfrac{A}{2}$ $C_1 = \dfrac{2[2T_3^2 - T_1(T_3-T_2)]}{AT_3}$ $C_2 = \dfrac{2T_1(T_3-T_2)}{AT_3}$ $C_3 = \dfrac{T_1T_2}{AT_3}$
$A\,\dfrac{1+sT_3}{1+sT_1+s^2T_1T_2}$ $T_2 > \dfrac{T_1}{4}\left(\text{complex roots}\right)$ $T_3 < T_1$	20 	$A = R_2$ $T_1 = 2R_1C_1 + R_2C_2$ $T_2 = \dfrac{R_1R_2C_1(C_1+2C_2)}{2R_1C_1+R_2C_2}$ $T_3 = 2R_1C_1$	$R_1 = \dfrac{AT_3^2}{4[T_1T_2 - T_3(T_1-T_3)]}$ $R_2 = A$ $C_1 = \dfrac{2[T_1T_2 - T_3(T_1-T_3)]}{AT_3}$ $C_2 = \dfrac{T_1-T_3}{A}$
	21 	$A = R_2$ $T_1 = \dfrac{C_1(2R_1C_2 + R_2C_1)}{2C_1+C_2}$ $T_2 = \dfrac{R_1R_2C_1C_2}{2R_1C_2+R_2C_1}$ $T_3 = \dfrac{2R_1C_1C_2}{2C_1+C_2}$	$R_1 = \dfrac{AT_3^2}{4[T_1T_2 - T_3(T_1-T_3)]}$ $R_2 = A$ $C_1 = \dfrac{2T_1T_2}{AT_3}$ $C_2 = \dfrac{4T_1T_2[T_1T_2 - T_3(T_1-T_3)]}{AT_3^2(T_1-T_3)}$
	22 	$A = R_3$ $T_1 = \dfrac{R_1(2R_2+R_3)C}{R_1+R_2}$ $T_2 = \dfrac{R_2R_3C}{2R_2+R_3}$ $T_3 = \dfrac{2R_1R_2C}{R_1+R_2}$	$R_1 = \dfrac{AT_3^2}{2[2T_1T_2 - T_3(T_1-T_3)]}$ $R_2 = \dfrac{AT_3}{2(T_1-T_3)}$ $R_3 = A$ $C = \dfrac{2T_1T_2}{AT_3}$

TABLE 3-2 (*Continued*)

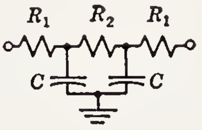
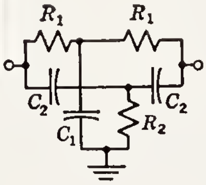
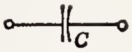
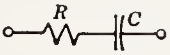
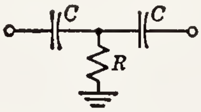
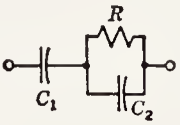
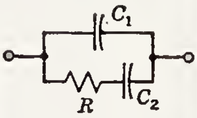
Transfer impedance function	Network	Relations	Inverse relations
$A(1 + sT_1)(1 + sT_2)$ $T_1 < T_2$	23 	$A = 2R_1 + R_2$ $T_1 = \frac{R_1R_2}{2R_1 + R_2} C$ $T_2 = R_1C$	$R_1 = A \frac{T_2 - T_1}{2T_2}$ $R_2 = A \frac{T_1}{T_2}$ $C = \frac{2T_2^2}{A(T_2 - T_1)}$
$A \frac{1 + sT_1}{1 + s^2T_1T_2}$	24 	$A = 2R_1$ $T_1 = \frac{R_1C_1}{2} = 2R_2C_2$ $T_2 = R_1C_2$	$R_1 = \frac{A}{2}$ $R_2 = \frac{AT_1}{4T_2}$ $C_1 = \frac{4T_1}{A}$ $C_2 = \frac{2T_2}{A}$
$\frac{1}{sB}$	25 	$B = C$	$C = B$
$\frac{1}{sB}(1 + sT)$	26 	$B = C$ $T = RC$	$R = \frac{T}{B}$ $C = B$
$\frac{1}{sB} \frac{1 + sT}{sT}$	27 	$B = \frac{C}{2}$ $T = 2RC$	$R = \frac{T}{4B}$ $C = 2B$
$\frac{1}{sB} \frac{1 + sT}{1 + s\theta T}$ $\theta < 1$	28 	$B = C_1$ $T = R(C_1 + C_2)$ $\theta = \frac{C_2}{C_1 + C_2}$	$R = \frac{T(1 - \theta)}{B}$ $C_1 = B$ $C_2 = \frac{B\theta}{1 - \theta}$
	29 	$B = C_1 + C_2$ $T = RC_2$ $\theta = \frac{C_1}{C_1 + C_2}$	$R = \frac{T}{B(1 - \theta)}$ $C_1 = B\theta$ $C_2 = B(1 - \theta)$

Table 3-2 (*Continued*)

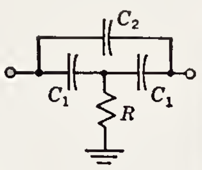
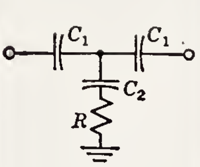
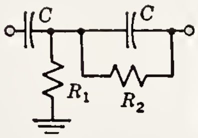
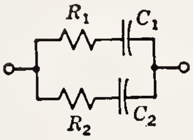
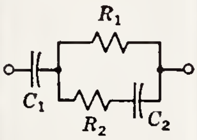
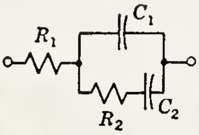
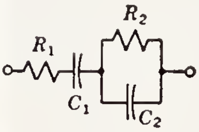
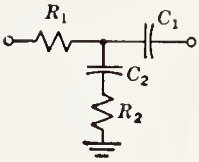
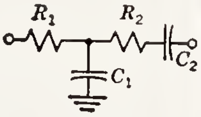
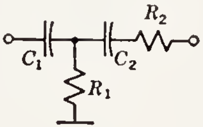
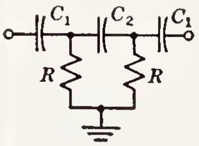
Transfer impedance function	Network	Relations	Inverse relations
$\frac{1}{sB}\frac{1+s\theta T}{1+sT}$ $\theta < 1$	30 	$B = C_2$ $T = \frac{2C_2 + C_1}{C_2} RC_1$ $\theta = \frac{2C_2}{2C_2 + C_1}$	$R = \frac{T\theta^2}{4B(1-\theta)}$ $C_1 = \frac{2B(1-\theta)}{\theta}$ $C_2 = B$
	31 	$B = \frac{C_1^2}{2C_1 + C_2}$ $T = RC_2$ $\theta = \frac{2C_1}{2C_1 + C_2}$	$R = \frac{T\theta^2}{4B(1-\theta)}$ $C_1 = \frac{2B}{\theta}$ $C_2 = \frac{4B(1-\theta)}{\theta^2}$
	32 	$B = \frac{R_1}{R_1 + R_2} C$ $T = R_2 C$ $\theta = \frac{2R_1}{R_1 + R_2}$	$R_1 = \frac{T\theta^2}{2B(2-\theta)}$ $R_2 = \frac{T\theta}{2B}$ $C = \frac{2B}{\theta}$
$\frac{1}{sB}\frac{(1+sT_1)(1+sT_3)}{1+sT_2}$ $T_1 < T_2 < T_3$	33 	$B = C_1 + C_2$ $T_1 = R_1 C_1$ $T_2 = (R_1 + R_2)\frac{C_1 C_2}{C_1 + C_2}$ $T_3 = R_2 C_2$	$R_1 = \frac{T_1(T_3 - T_1)}{B(T_2 - T_1)}$ $R_2 = \frac{T_3(T_3 - T_1)}{B(T_3 - T_2)}$ $C_1 = \frac{B(T_2 - T_1)}{T_3 - T_1}$ $C_2 = \frac{B(T_3 - T_2)}{T_3 - T_1}$
	34 	$B = C_1$ $T_2 = (R_1 + R_2)C_2$ $T_1 T_3 = R_1 R_2 C_1 C_2$ $T_1 + T_3 = R_1 C_1 + R_2 C_2 + R_1 C_2$	$R_1 = \frac{T_1 + T_3 - T_2}{B}$ $R_2 = \frac{T_1 T_3 (T_1 + T_3 - T_2)}{B(T_3 - T_2)(T_2 - T_1)}$ $C_1 = B$ $C_2 = \frac{B(T_3 - T_2)(T_2 - T_1)}{(T_1 + T_3 - T_2)^2}$
	35 	$B = C_1 + C_2$ $T_2 = R_2 \frac{C_1 C_2}{C_1 + C_2}$ $T_1 T_3 = R_1 R_2 C_1 C_2$ $T_1 + T_3 = R_1 C_1 + R_2 C_2 + R_1 C_2$	$R_1 = \frac{T_1 T_3}{B T_2}$ $R_2 = \frac{(T_1 T_2 + T_2 T_3 - T_1 T_3)^2}{B T_2 (T_3 - T_2)(T_2 - T_1)}$ $C_1 = \frac{B T_2^2}{T_1 T_2 + T_2 T_3 - T_1 T_3}$ $C_2 = \frac{B(T_3 - T_2)(T_2 - T_1)}{T_1 T_2 + T_2 T_3 - T_1 T_3}$

TABLE 3-2 (*Continued*)

Transfer impedance function	Network	Relations	Inverse relations
$\frac{1}{sB}\frac{(1+sT_1)(1+sT_3)}{1+sT_2}$ $T_1 < T_2 < T_3$	36  	$B = C_1$ $T_2 = R_2C_2$ $T_1T_3 = R_1R_2C_1C_2$ $T_1 + T_3 = R_1C_1 + R_2C_2 + R_2C_1$	$R_1 = \frac{T_1T_3}{BT_2}$ $R_2 = \frac{(T_3 - T_2)(T_2 - T_1)}{BT_2}$ $C_1 = B$ $C_2 = \frac{BT_2^2}{(T_3 - T_2)(T_2 - T_1)}$
	37  	$B = C_1$ $T_2 = R_2C_2$ $T_1T_3 = R_1R_2C_1C_2$ $T_1 + T_3 = R_1C_1 + R_2C_2 + R_1C_2$	$R_1 = \frac{T_1T_3}{BT_2}$ $R_2 = \frac{T_1T_2T_3}{B(T_3 - T_2)(T_2 - T_1)}$ $C_1 = B$ $C_2 = \frac{B(T_3 - T_2)(T_2 - T_1)}{T_1T_3}$
$\frac{1}{sB}(1+sT_1)(1+sT_2)$ $T_1 \neq T_2$	38  	$B = C_2$ $T_1T_2 = R_1R_2C_1C_2$ $T_1 + T_2 = R_1C_1 + R_2C_2 + R_1C_2$	$R_1 = \frac{(\sqrt{T_1} - \sqrt{T_2})^2}{B}$ $R_2 = \frac{\sqrt{T_1T_2}}{B}$ $C_1 = \frac{B\sqrt{T_1T_2}}{(\sqrt{T_1} - \sqrt{T_2})^2}$ $C_2 = B$
$\frac{1}{sB}\frac{(1+sT_1)(1+sT_2)}{s\sqrt{T_1T_2}}$ $T_1 \neq T_2$	39  	$B = C_2$ $T_1T_2 = R_1R_2C_1C_2$ $T_1 + T_2 = R_1C_1 + R_2C_2 + R_1C_2$	$R_1 = \frac{(\sqrt{T_1} - \sqrt{T_2})^2}{B}$ $R_2 = \frac{\sqrt{T_1T_2}}{B}$ $C_1 = \frac{B\sqrt{T_1T_2}}{(\sqrt{T_1} - \sqrt{T_2})^2}$ $C_2 = B$
$\frac{1}{sB}\frac{(1+sT_1)(1+sT_2)}{s^2T_1T_2}$ $T_1 < T_2$	40  	$B = \frac{C_1C_2}{C_1 + 2C_2}$ $T_1 = RC_1$ $T_2 = R(C_1 + 2C_2)$	$R = \frac{T_1(T_2 - T_1)}{2BT_2}$ $C_1 = \frac{2BT_2}{T_2 - T_1}$ $C_2 = \frac{BT_2}{T_1}$

ance function. Table 3-2 is a compilation of the transfer impedance functions for a number of simple networks. We shall be primarily interested in the case where these networks are used either as the input or as the feedback of a high-gain amplifier. In this case the transfer function of the operational amplifier will be, by Eq. (1-15),

$$\frac{E_o}{E_i} = -\frac{Z_f}{Z_i} \tag{3-30}$$

where Z_i and Z_f are the transfer impedance functions of the input and feedback networks, respectively. Thus we can simulate by this method any transfer function $F(s)$ which can be written in the form

$$F(s) = \frac{A(s)}{B(s)} \tag{3-31}$$

where $A(s)$ and $B(s)$ are the transfer functions of networks given in Table 3-2. The following examples will illustrate the use of Table 3-2.

Example 1. It is desired to simulate the transfer function

$$\frac{E_o}{E_i} = k\frac{1+3s}{1+s}$$

Looking in the table of transfer impedance functions, we note that network 2 of Table 3-2 has the transfer impedance function

$$Z = \frac{A}{1+Ts}$$

with $A = R$ and $T = RC$. If we use this circuit as both the input and the feedback transfer impedance function, the desired transfer function will be simulated. Therefore make

$$Z_f = \frac{R_f}{1+R_fC_fs} \qquad Z_i = \frac{R_i}{1+R_iC_is}$$

Then
$$\frac{E_o}{E_i} = -\frac{R_f}{R_i}\frac{1+R_iC_is}{1+R_fC_fs}$$

To satisfy our particular values, let

$R_i = 3$ megohms $\qquad C_i = 1\ \mu$f
$R_f = 1$ megohm $\qquad C_f = 1\ \mu$f

The transfer function of the operational amplifier is then

$$\frac{E_o}{E_i} = -\frac{1}{3}\frac{1+3s}{1+s}$$

This circuit is shown as Fig. 3-14*a*.

An alternative realization of this transfer function can be obtained by considering network 4 of Table 3-2. This network has the transfer impedance function

$$Z = A\,\frac{1 + s\theta T}{1 + sT}$$

where

$$\theta = \frac{R_1}{R_1 + R_2}$$

$$A = R_1 + R_2 \qquad \text{and} \qquad T = R_2C$$

Noting that $\theta < 1$, if we use this circuit in the input to an amplifier and use a resistance feedback, we can simulate the desired transfer function. In this case

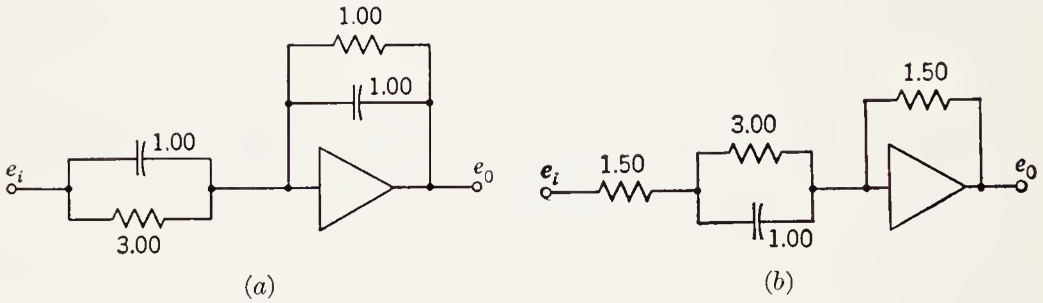


FIG. 3-14. (a) A circuit to simulate the transfer function

$$\frac{E_o}{E_i} = -\frac{1}{3}\frac{1 + 3s}{1 + s}$$

(b) Alternative circuit for simulating

$$\frac{E_o}{E_i} = -\frac{1}{3}\frac{1 + 3s}{1 + s}$$

$$\theta = \frac{R_1}{R_1 + R_2} = \frac{1}{3}$$

Let $C = 1.0$ μf; then $R_2 = 3.0$ megohms, $R_1 = 1.5$ megohms, and $A = R_1 + R_2 = 4.5$. The feedback resistance is given by $R = 4.5k$, where k is the desired gain. The circuit for a gain of ⅓ is given in Fig. 3-14*b*. This latter circuit has the advantage over that of Fig. 3-14*a* since it requires the use of only one capacitor and since the gain of the over-all circuit is adjustable.

Example 2. The Imperfect Differentiator. It is sometimes desirable to simulate a function which has the property of differentiation over a limited frequency range without the disadvantage of noise due to the differentiator characteristic. This can be accomplished by running the differentiator through a low-pass filter, i.e., by replacing the differentiator by a circuit with the transfer function $E_o/E_i = s/(1 + Ts)$, with $T \ll 1$. Figure 3-15 shows a circuit with the transfer function

$$\frac{E_o}{E_i} = -\frac{R_2Cs}{1 + R_1Cs} \tag{3-32}$$

If we make $C = 1\ \mu f$, $R_2 = 1$ megohm, $R_1 = 0.01$ megohm, then

$$\frac{E_o}{E_i} = \frac{-s}{1 + 0.01s} \tag{3-33}$$

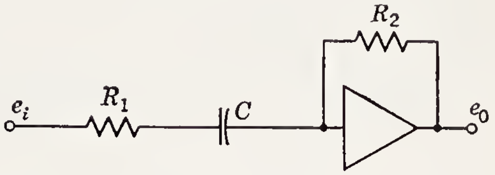


FIG. 3-15. An imperfect differentiator circuit.

The low-pass filter of this circuit then would have a break frequency of 16 cps.

Example 3. Double Integration on One Amplifier. Utilizing the network methods illustrated in this section, it is possible to perform a double integration on a single amplifier. For instance, consider network 27 of Table 3-2. This network has the transfer impedance function

$$\frac{1 + Ts}{TBs^2}$$

Thus, if we use this network in the feedback and network 3 in the input, the transfer function for the resulting operational amplifier is

$$\frac{E_o}{E_i} = -\frac{K}{s^2} \tag{3-34}$$

provided that the first-order lag terms cancel. This circuit is illustrated in Fig. 3-16. Using the notation of this figure, Eq. (3-34) becomes

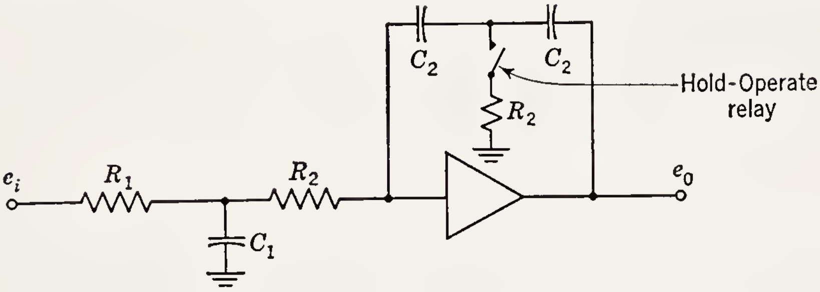


FIG. 3-16. Circuit for double integration of one amplifier.

$$\frac{E_o}{E_i} = -\frac{1}{2R_1R_2C_2{}^2s^2} \tag{3-35}$$

provided that

$$\frac{R_1}{R_2} = 4\frac{C_2}{C_1} \tag{3-36}$$

Equation (3-36) guarantees that the lag terms cancel.

The only unusual feature in this circuit occurs if one desires to apply an initial condition. As the circuit was derived, any charge applied to the feedback capacitances will bleed off through the resistance R_2. To allow the application of initial conditions, it is necessary to put a hold-operate relay* in the path to ground. This relay is closed only when the computer control is in the "operate" position. Hence the path to ground is open until the start of the computation and the initial condition is preserved.

Example 4. The One-amplifier Oscillator. Using the circuit for double integration on one amplifier, it is possible to set up a one-amplifier linear-oscillator circuit. Since the differential equation of the oscillator is

$$\ddot{X} = -\omega_n^2 X$$

we have from Eq. (3-35) that the frequency of oscillation is given by

$$\omega_n = \frac{1}{\sqrt{2R_1R_2C_2^2}}$$

provided that $R_1/R_2 = 4(C_2/C_1)$ is observed. The circuit is given in Fig. 3-17. Since there is no way to set an initial condition on velocity, the initial condition set on the amplifier determines the magnitude of the oscillation [that is, $X(0) = A$]. To allow the introduction of initial conditions, the computer relays must be used in the same fashion as in the previous example.

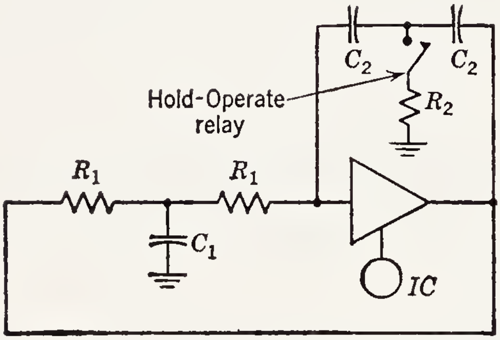


FIG. 3-17. The one-amplifier oscillator circuit.

Example 5. A Second-order Lag Circuit. If we note that the transfer function of a second-order lag can be written as

$$\frac{1}{1 + (2\xi/\omega_n)s + s^2/\omega_n^2} = \frac{1 + T_1s}{1 + T_1s + T_1T_2s^2}\,\frac{1}{1 + T_1s} \tag{3-37}$$

it is obvious that such a function can be simulated using Table 3-2. If we use network 3 in the input and network 15 in the feedback of a high-gain amplifier, we have satisfied the relationships of Eq. (3-37), provided that

$$R_1C_1 = 4R_2C_2 \tag{3-38}$$

This circuit is shown as Fig. 3-18. The technique can be expanded to the simulation of a third-order lag by using network 23 in the input. The circuit is illustrated in Fig. 3-20.

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* For a discussion of the functioning of the computer control relays, see Chap. 8.

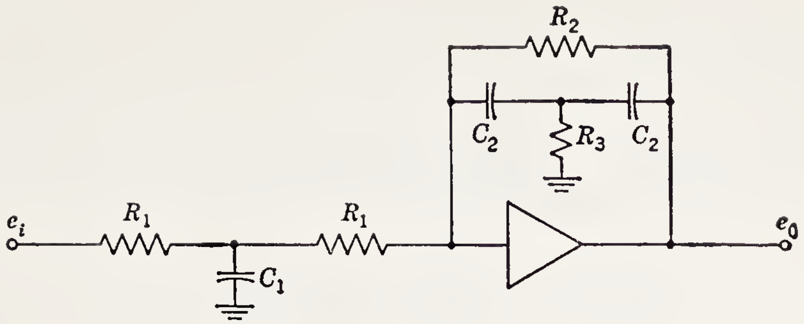


FIG. 3-18. A second-order lag circuit.

3-7. COMPARISON OF METHODS

The previous sections have provided a framework of techniques for analysis of linear systems and for programming these systems on an analog computer. In Chap. 1 mention was made of two philosophies of patch-board design. In this section we shall consider the impact of these philosophies upon equipment utilization, the relative merits of the various types of programming, and their compatibility with patch-board design.

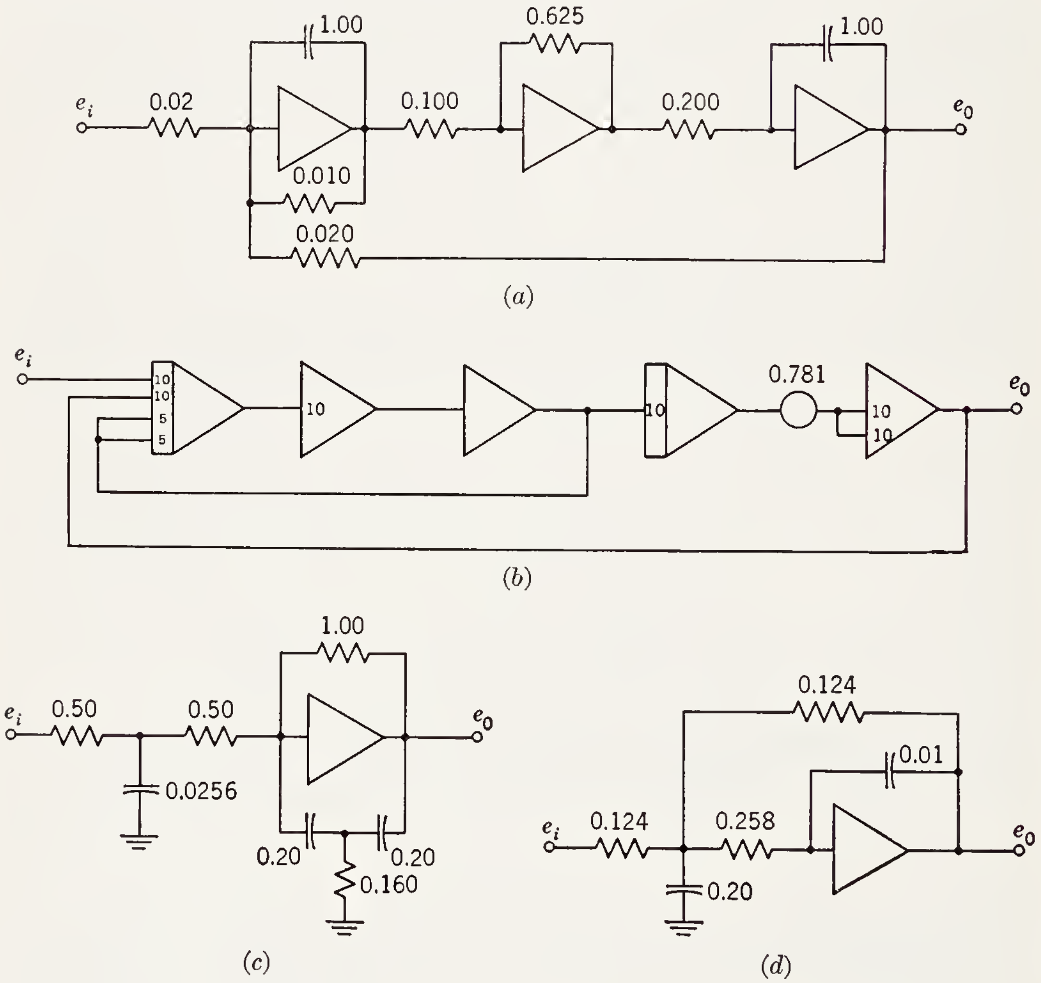
As more and more computers change to fixed patch-board gains and operations, it becomes increasingly difficult to achieve good equipment utilization. In order to facilitate greater precision in analog computing, the computing capacitors and resistors are being enclosed in ovens to maintain close temperature tolerances. This high degree of accuracy is essential for the solution of extremely large or critical problems, but for a myriad of problems in the engineering field it is entirely superfluous. For this latter field of problems 1 per cent components are satisfactory, and there is no need for temperature control. The network methods described in Sec. 3-6 are most applicable to the latter type of problem and therefore to those machines which do not employ standard gains. The method may also be employed to meet more accurate requirements if sufficient high-precision passive elements are available.

One of the easiest ways to reduce the number of amplifiers required for the solution of a linear differential equation is the substitution of network methods for the direct solution of the differential equation by a number of separate summations and integrations. The use of networks requires an increase in passive elements, often of unusual values, and generally results in a loss of flexibility. However, when amplifiers are at a premium, these problems can usually be tolerated. In addition there are often portions of a problem where flexibility is unnecessary, an example being the transfer function of a known, fixed piece of hardware. The following examples, which were taken from actual engineering problems, will serve to indicate the savings in equipment which can be realized by the use of network methods.

Example 1. Simulation of a Rate Gyro. The first example we shall consider is the simulation of the dynamic response of a rate gyro with a second-order lag circuit. Assuming a natural frequency of 20 cps and a damping ratio of 0.4, we have the transfer function

$$\frac{\dot{\theta}_o}{\dot{\theta}_i} = \frac{1}{1 + 0.0064s + 0.000064s^2} \tag{3-39}$$

Figure 3-19*a* and *b* shows two circuits that were obtained by direct solution of the differential equation. The first circuit utilizes gains that are not usually available on standardized computers, while the second circuit could be put on any machine. In Fig. 3-19*c* and *d* two other ways



(*a*)

(*b*)

(*c*) (*d*)

FIG. 3-19. Four methods for simulating the transfer function

$$\frac{E_o}{E_i} = \frac{-1}{1 + 0.0064s + 0.000064s^2}$$

of realizing this transfer function are shown. Figure 3-19*d* utilizes a circuit discussed at length in Sec. 3-5. The circuit of Fig. 3-19*c* was described in Sec. 3-6.

Example 2. Simulation of a Servomechanism. The transfer function of a particular servomechanism can be approximated by

$$\frac{E_o}{E_i} = \frac{1}{(1 + 0.1s)(1 + 0.004s + 0.0004s^2)} \tag{3-40}$$

The three circuits of Fig. 3-20 illustrate methods of simulating this transfer function.

As can be seen from the previous examples, the direct solution of the differential equation requires the use of many more amplifiers than would be necessary using network methods. This is especially true for those

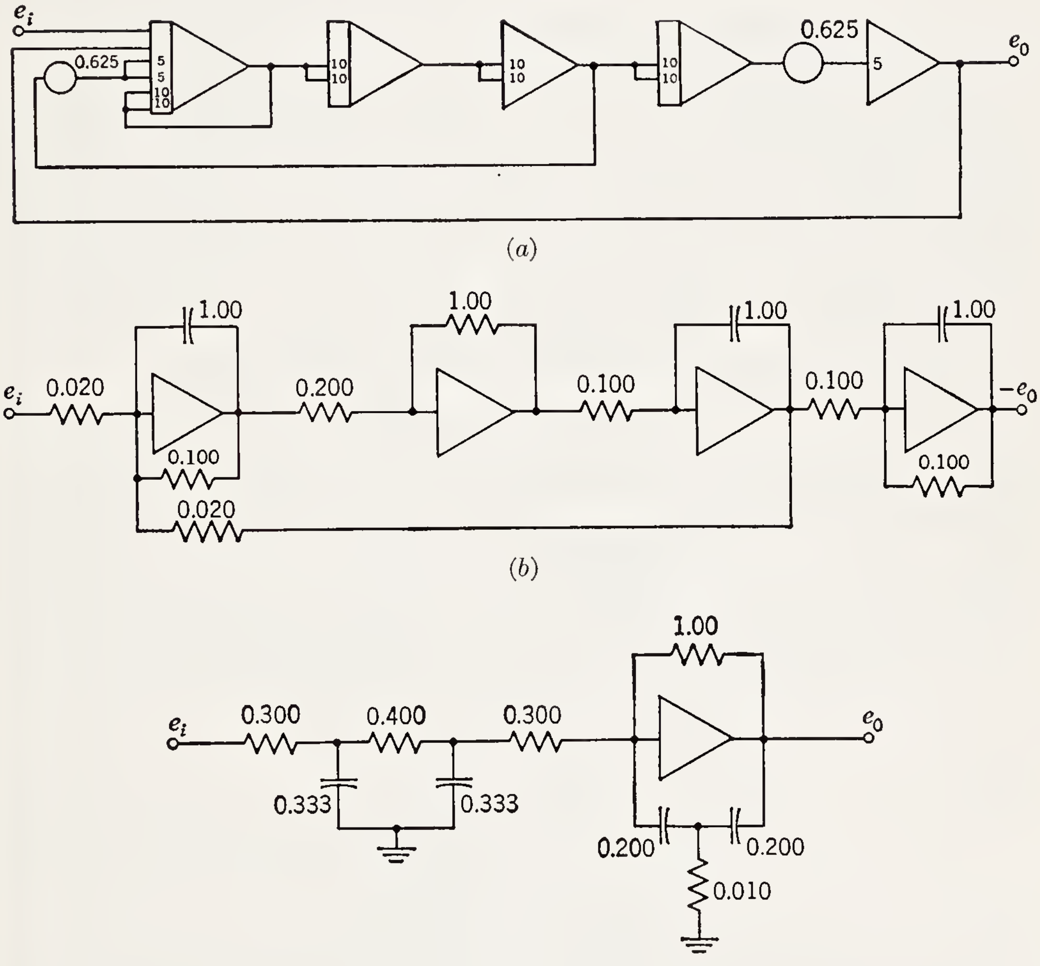


FIG. 3-20. Three circuits for simulating the transfer function

$$\frac{E_o}{E_i} = \frac{-1}{(1 + 0.1s)(1 + 0.004s + 0.0004s^2)}$$

computers having fixed, standard patch-board gains. In the case of these computers the amount of equipment necessary is governed by the maximum gain available in each amplifier. A further problem arises when direct-solution methods are used. Since a number of amplifiers are connected into a closed loop, the possibility of amplifier characteristics affecting the result is greatly increased.* On the credit side the direct-solution method is much more straightforward and is definitely more applicable to fixed-gain computers.

It is impossible to state generally which method is best, even for a given problem. It depends upon the experience of the operator, the flexibility desired, the availability of equipment, and many other factors. The operator must exercise his own judgment as to which method is most applicable.

3-8. THE SIMULATION OF DEAD TIME

In physical problems one often encounters the problem of simulating or describing two functions identical to one another except for a shift in time. When such a time shift occurs, the system is said to contain "dead time," or "delay time." Dead time can be expressed by the following mathematical relationship: Given $f(t)$ for $t \geq 0$, we say that the function $f(t - \tau)$ differs from the function $f(t)$ by the dead time τ, where we define $f(t - \tau)$ to be zero for $t < \tau$.

An important theorem of Laplace transforms states that

$$\mathcal{L}[f(t-\tau)] = e^{-\tau s}\mathcal{L}[f(t)]$$

and so the transfer function of dead time is

$$\frac{\mathcal{L}[f(t-\tau)]}{\mathcal{L}[f(t)]} = e^{-\tau s} = \frac{E_o}{E_i} \tag{3-41}$$

Substituting $s = j\omega$ to find the frequency response gives

$$\frac{E_o}{E_i} = e^{-j\tau\omega} = \cos\tau\omega - j\sin\tau\omega$$

Thus

$$\left|\frac{E_o}{E_i}\right| = \sqrt{\cos^2\tau\omega + \sin^2\tau\omega} = 1 \tag{3-42}$$

$$\phi = \arctan\left(-\frac{\sin\tau\omega}{\cos\tau\omega}\right) = -\tau\omega \tag{3-43}$$

or the frequency transfer function of dead time is characterized by an amplitude ratio which is unity for all frequencies and a phase angle which varies linearly with frequency over the desired range of interest.

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* See the discussions on circuit stability given in Chaps. 6 and 8.

Since $e^{-\tau s}$ cannot be expressed as a rational fraction, the transfer function for dead time can be simulated only approximately. Many methods have been developed during the past few years to approximate this transfer function. Some of these methods involve the use of electromechanical devices such as tape recorders, and these methods simply insert a device having the property of dead time into the circuit. These techniques appear to hold the most promise, but all require additional equipment not generally found in the analog facility. The remaining methods depend on an approximation of the transfer function $e^{-\tau s}$ by an appropriate series or technique of network synthesis. These methods give a satisfactory approximation only over a limited frequency range. These attempts can be generally grouped into two approaches:

1. Fit the amplitude exactly, and then take the best phase fit obtainable under those circumstances.
2. Fit both the amplitude and the phase approximately.

The first approach stems from the fact that the quotient of any pair of complex-conjugate polynomials has an amplitude ratio of 1 and also possesses considerable phase shift. Therefore the problem of approximation is reduced to finding a transfer function which is the ratio of complex-conjugate polynomials whose phase varies linearly over some specified frequency range. The simplest such term is

$$\frac{E_o}{E_i} = \frac{\tau_1^2(\tau s)^2 - 2\xi_1\tau\tau_1 s + 1}{\tau_1^2(\tau s)^2 + 2\xi_1\tau\tau_1 s + 1} \qquad (3\text{-}44)$$

and suitable values of ξ_1 and τ_1 are selected to make the phase vary linearly over a frequency range of interest. The most common method using this technique employs the so-called "Pade" approximation. This method is discussed in detail in many references,* and so we shall not present the expansion here. This method yields what may be called optimally flat phase characteristics at the origin but does not necessarily fit the phase characteristics as well over an extended frequency range. For example, the second-order Pade approximation yields

$$e^{-\tau s} \doteq \frac{(\tau s)^2 - 6\tau s + 12}{(\tau s)^2 + 6\tau s + 12} \qquad (3\text{-}45)$$

while a simple curve fitting by the authors yields

$$e^{-\tau s} \doteq \frac{(\tau s)^2 - 4.7\tau s + 10}{(\tau s)^2 + 4.7\tau s + 10} \qquad (3\text{-}46)$$

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* For example, see C. Johnson, "Analog Computer Techniques," McGraw-Hill Book Company, Inc., New York, 1956; and C. D. Morrill, A Sub-audio Time Delay Circuit, *IRE, Trans. on Electronic Computers*, June, 1954, pp. 45–49.

The phase characteristics of Eqs. (3-45) and (3-46) are plotted in Fig. 3-21. From this figure it can be seen that the Pade approximation has a phase error of greater than 10 deg for $\omega\tau > 3$, while Eq. (3-46) is within 10 deg of the desired phase for $\omega\tau \leq 3.9$. Thus the frequency range of the approximation can be extended 30 per cent with little or no detrimental effect to the approximation of the rest of the curve. If higher frequency response is desired, it is necessary to add more terms. Again, a better fit of the phase characteristics can be obtained by curve-fitting

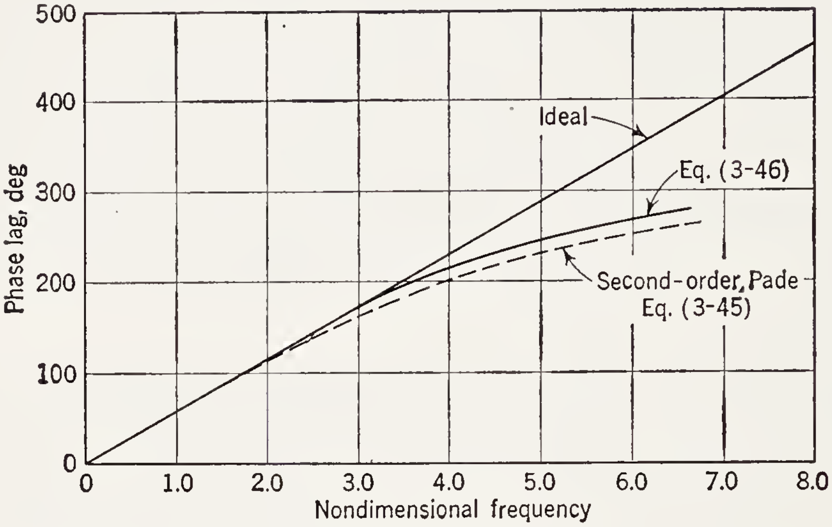

FIG. 3-21. Phase characteristic for second-order dead-time approximation.

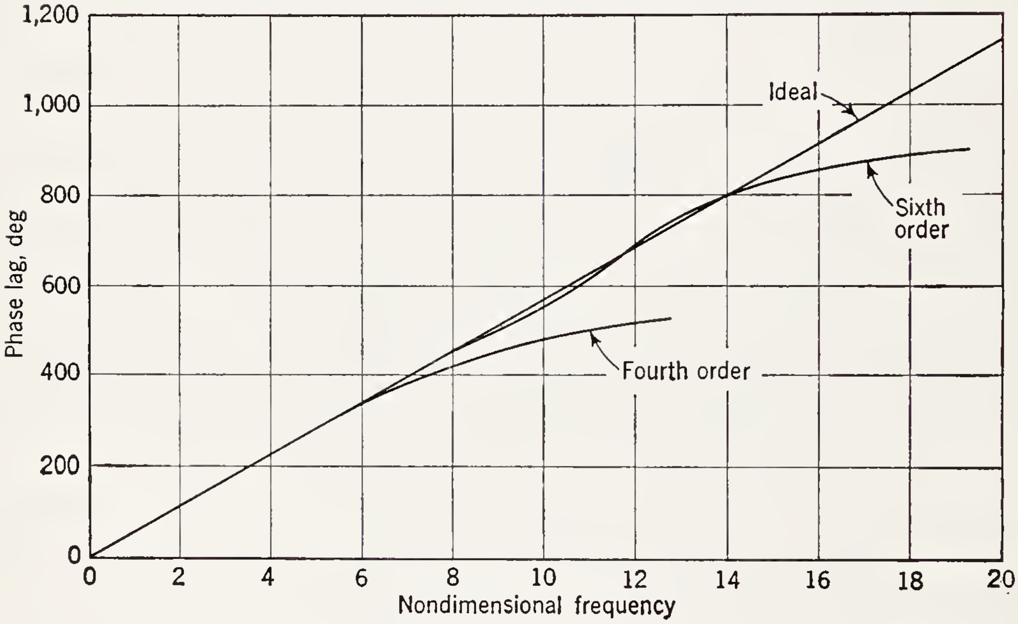

FIG. 3-22. Phase characteristic for higher-order dead-time approximation.

methods than by the Pade series.* The equations

$$F(s) = \frac{(\tau s)^2 - 11\tau s + 36}{(\tau s)^2 + 11\tau s + 36} \, \frac{(\tau s)^2 - 8.42\tau s + 46.0}{(\tau s)^2 + 8.42\tau s + 46.0} \qquad (3\text{-}47)$$

and

$$F(s) = \frac{(\tau s)^2 - 13.5\tau s + 50}{(\tau s)^2 + 13.5\tau s + 50} \, \frac{(\tau s)^2 - 8.42\tau s + 46.0}{(\tau s)^2 + 8.42\tau s + 46.0} \, \frac{(\tau s)^2 - 4.5\tau s + 144}{(\tau s)^2 + 4.5\tau s + 144} \qquad (3\text{-}48)$$

represent approximations to dead-time characteristics which were obtained by curve-fitting methods. The phase characteristics of these curves are shown in Fig. 3-22. The major drawback to the use of complex-conjugate polynomials to simulate dead time is the response of this approximation to a step input or other input having high-frequency components. Since the amplitude ratio is always 1 but the phase is approximated only for low frequencies, the output "rings" badly for high-frequency inputs. This behavior is illustrated in Fig. 3-23, which shows the response of Eq. (3-47) to sine wave, triangular wave, and step inputs.

This problem can be alleviated through the application of the second approach mentioned previously. That is, if the amplitude is attenuated at high frequencies, the ringing will be decreased. Thus this second technique essentially calls for the design of an ideal low-pass filter. A simple

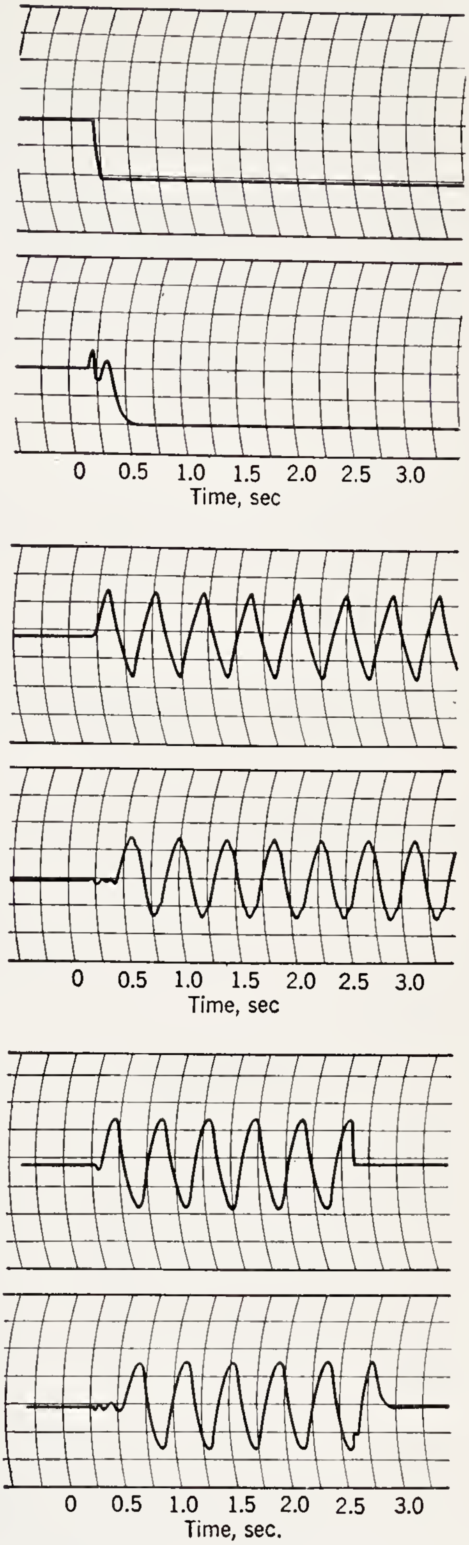

Fig. 3-23. Transient response of a complex-conjugate dead-time approximation.

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* For discussion, see letter by W. E. Thomson, in correspondence, *IRE, Trans. on Electronic Computers*, June, 1955, p. 21.

application of this method uses a combination of a complex-conjugate approximation with a low-pass filter consisting of first- and/or second-order lag terms. The technique is to pick a complex-conjugate approximation whose phase characteristic falls slightly below the ideal at low frequencies. Lag terms are then added as necessary to bring the phase to the ideal (for a limited frequency range) and also to attenuate the amplitude at high frequencies. This method thus preserves the low-frequency characteristics of the complex-conjugate method and improves upon the high-frequency characteristics. It should be pointed out, however, that the ringing cannot be completely eliminated without greatly limiting the frequency range of the approximation. This is true, since our low-pass filter is far from ideal, and results in considerable amplitude attenuation near and below the break frequency. The following expressions illustrate two approximations using this technique:

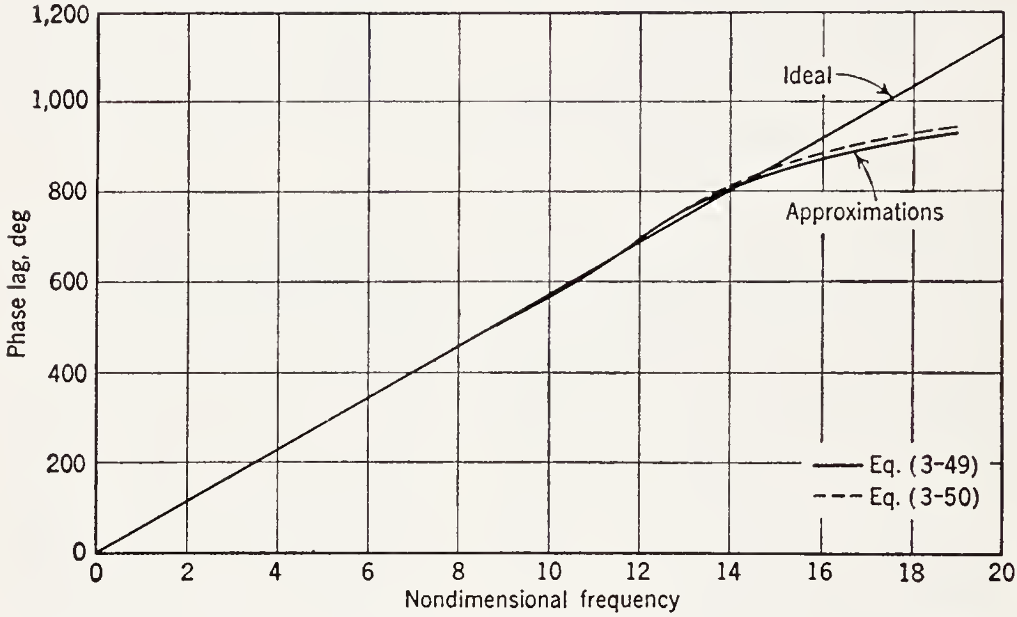

FIG. 3-24. Phase characteristic for dead-time approximation with amplitude attenuation.

$$\frac{(\tau s)^2 - 17\tau s + 72.2}{(\tau s)^2 + 17\tau s + 72.2}\,\frac{(\tau s)^2 - 8.42\tau s + 46}{(\tau s)^2 + 8.42\tau s + 46}\,\frac{(\tau s)^2 - 4.97\tau s + 150.6}{(\tau s)^2 + 4.97\tau s + 150.6}\,\frac{1}{1 + 0.0635\tau s} \qquad (3\text{-}49)$$

$$\frac{(\tau s)^2 - 17\tau s + 72.2}{(\tau s)^2 + 17\tau s + 72.2}\,\frac{(\tau s)^2 - 8.42\tau s + 46}{(\tau s)^2 + 8.42\tau s + 46}\,\frac{(\tau s)^2 - 4.97\tau s + 150.6}{(\tau s)^2 + 4.97\tau s + 150.6}\left(\frac{1}{1 + 0.02\tau s}\right)^3 \qquad (3\text{-}50)$$

The phase characteristics of these equations are shown in Fig. 3-24. The amplitude characteristics of these functions are, of course, the same as those of $1/(1 + 0.0635\tau s)$ and $[1/(1 + 0.02\tau s)]^3$, respectively.

The previous discussion has indicated that an approximation to dead time may be realized using terms of the form

$$\frac{E_o}{E_i} = \frac{(\tau s)^2/\omega_n^2 - (2\xi/\omega_n)\tau s + 1}{(\tau s)^2/\omega_n^2 + (2\xi/\omega_n)\tau s + 1} \tag{3-51}$$

We wish now to consider computer circuits for simulating this transfer function. To illustrate one such circuit, consider the function

$$F(s) = \frac{1 - as + bs^2}{1 + as + bs^2}$$

This can be written

$$F(s) = 1 - \frac{2as}{1 + as + bs^2} \tag{3-52}$$

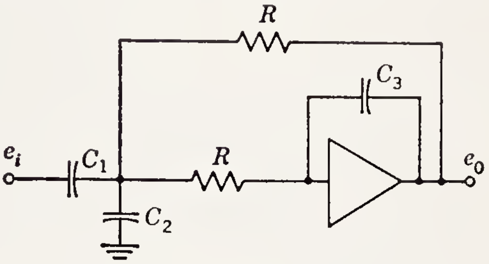

FIG. 3-25. A circuit for simulating the transfer function

$$\frac{E_o}{E_i} = \frac{-ks}{1 + as + bs^2}$$

Now in Sec. 3-5 the five-impedance network shown as Fig. 3-9 was examined in some detail. If the impedances are shown in Fig. 3-25, the transfer function of the circuit has the form

$$\frac{E_o}{E_i} = \frac{-RC_1s}{1 + 2RC_3s + R^2C_3(C_1 + C_2)s^2} \tag{3-53}$$

This circuit thus yields an output similar in form to the last term of Eq. (3-52). A continuation of this line of reasoning leads us to the circuit of Fig. 3-26, which can be used to simulate the transfer function of Eq. (3-51). The relationships for this circuit can be expressed as

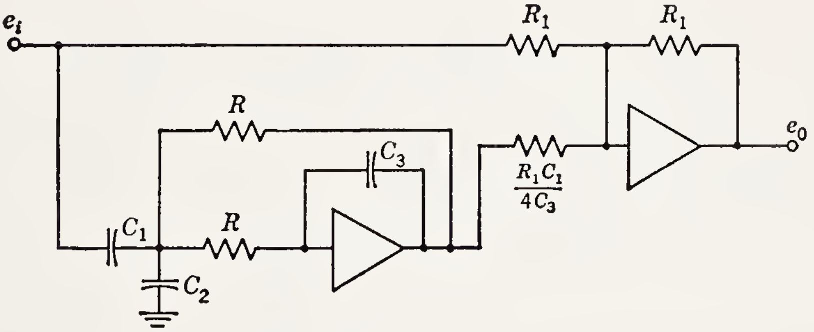

FIG. 3-26. A circuit for the simulation of dead time.

functions of the terms of Eq. (3-51) as

$$
\begin{aligned}
C_3(C_1 + C_2)R^2 &= \frac{\tau^2}{\omega_n{}^2} \\
RC_3 &= \frac{\xi_n \tau}{\omega_n}
\end{aligned}
\qquad (3\text{-}54)
$$

This can be written in terms of ξ_n and ω_n as

$$\xi_n = \sqrt{\frac{C_3}{C_1 + C_2}} \qquad (3\text{-}55)$$

$$\omega_n = \frac{\tau/R}{\sqrt{C_3(C_1 + C_2)}} \qquad (3\text{-}56)$$

Equation (3-56) shows that once a circuit has been set up for a given ξ_n and ω_n any change in R can be thought of as giving us a proportional change in τ. Thus we can vary the length of the dead time by varying the value of R.

One advantage of this method of simulation of dead time is that any number of such circuits can be cascaded to give higher-order representations. It also lends itself to the approach utilizing amplitude attenuation at high frequency. This use of the circuit will be discussed later. The following example shows an application of two such circuits to give a fourth-order complex-conjugate approximation of dead time.

Example. An approximation to dead time good for frequencies less than about $\omega\tau = 6.5$ is given by Eq. (3-47). If $\tau = 0.2$ sec, this approximation is valid for $\omega < 6$ cps. Substituting $\tau = 0.2$ sec into Eq. (3-47) and normalizing, we have

$$e^{-0.2s} \doteq \frac{0.001112s^2 - 0.0612s + 1}{0.001112s^2 + 0.0612s + 1} \, \frac{0.00087s^2 - 0.0364s + 1}{0.00087s^2 + 0.0364s + 1} \qquad (3\text{-}57)$$

Using the circuit of Fig. 3-26 to simulate the first pair of complex-conjugate components, we get the equations

$$R^2C_3(C_1 + C_2) = 0.001112$$

and

$$2RC_3 = 0.0612$$

If we let $C_3 = 0.1\ \mu\text{f}$, then $R = 0.306$ megohm and

$$C_1 + C_2 = 0.119\ \mu\text{f}$$

We can satisfy this equation quite closely if we let $C_1 = 0.1\ \mu\text{f}$ and

$C_2 = 0.02\ \mu\text{f}$. If we let $R_1 = 1.0$ megohm, we have that

$$\frac{R_1C_1}{4C_3} = 0.25 \text{ megohm}$$

Using the same circuit on the second pair, we obtain the equations

$$\begin{aligned} R^2C_3(C_1 + C_2) &= 0.00087 \\ 2RC_3 &= 0.0364 \end{aligned}$$

If we let $C_3 = 0.05\ \mu\text{f}$, then $R = 0.364$ megohm and

$$C_1 + C_2 = 0.131\ \mu\text{f}$$

Let $C_1 = 0.10\ \mu\text{f}$ and $C_2 = 0.031\ \mu\text{f}$. If we let $R_1 = 1.0$ megohm, then $R_1C_1/4C_3 = 0.50$ megohm. The circuit resulting from these computations is given in Fig. 3-27.

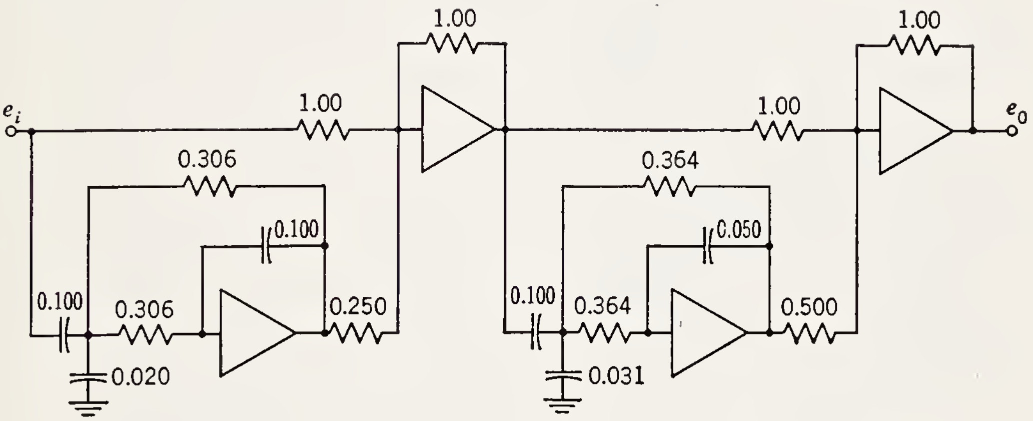

FIG. 3-27. Dead-time circuit for $\tau = 0.20$ sec.

If we again consider the circuit of Fig. 3-26, we see that this circuit can be modified to give a different form of approximation. For if a capacitance is placed in parallel with the feedback resistance R of the output amplifier, the transfer function of the circuit is multiplied by a first-order lag term. Thus it is possible to attenuate the amplitude at high frequencies without the use of any additional amplifiers. Because of the similarity of this technique to those already illustrated, no examples will be given.

The circuit of Fig. 3-25 can also be cascaded in a different manner to yield higher-order approximations to the transfer function of dead time. Figure 3-28 shows a means of paralleling these circuits to obtain a sixth-order approximation utilizing amplitude attenuation at high frequencies. The transfer function used is that of Eq. (3-49) with $\tau = 0.10$ sec. This technique can be extended to higher-order approximation but becomes quite unwieldy when this is done. The proof that this circuit satisfies the transfer function of Eq. (3-49) is left for the reader to perform as a problem.

It is recognized that, in many computer installations, the values of the resistances and capacitances required to realize the previous dead-time circuits are not readily available. Therefore it is desirable to have available an alternative circuit that can be realized on those computer installations where the values of the resistors and capacitors cannot be readily altered. The technique we shall use to derive the desired circuit is the *M* method introduced in Sec. 2-9.

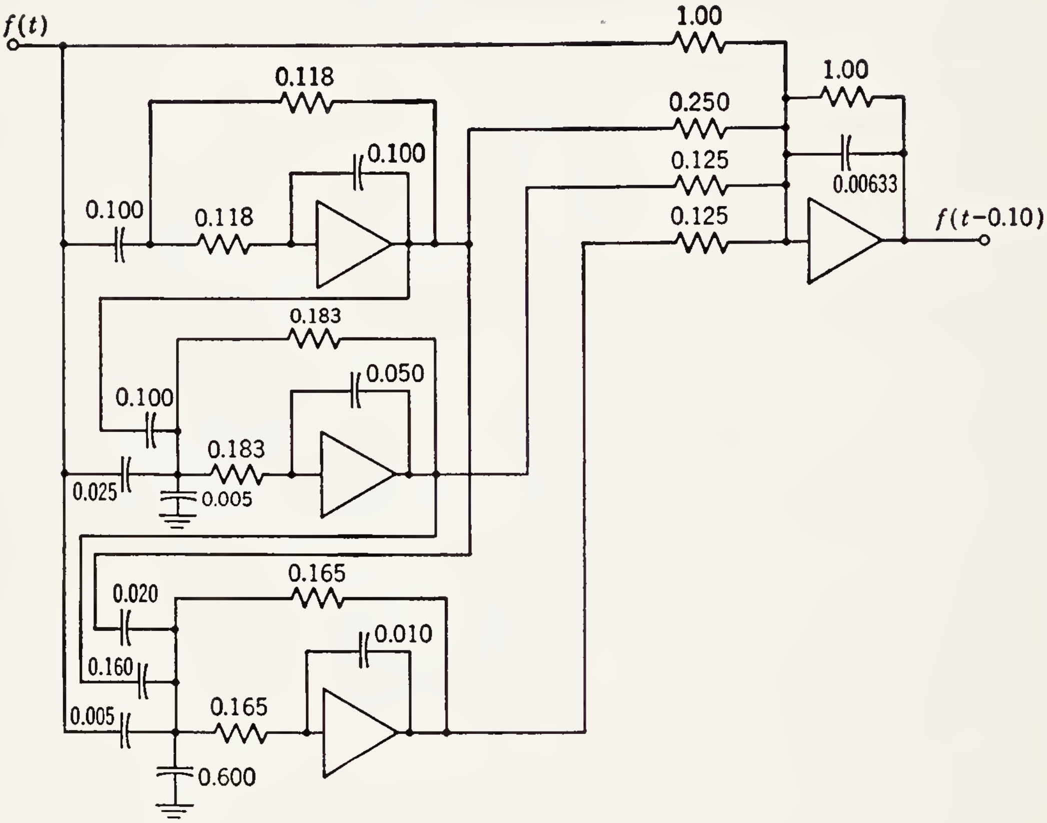

FIG. 3-28. Alternative circuit for the approximation of dead time.

This technique can be used to derive a circuit for any transfer function and can, therefore, be used with any of the methods discussed previously. As an example we shall consider the application of this method to the sixth-order complex-conjugate polynomial approximation which was given as Eq. (3-48).

From Eq. (3-48) a sixth-order approximation to dead time with

$\tau = 0.40$ sec is given by

$$\frac{E_o}{E_i} = \frac{0.004096s^6 - 0.2705s^5 + 11.83s^4 - 329.1s^3 + 5{,}950s^2 - 64{,}160s + 331{,}200}{0.004096s^6 + 0.2705s^5 + 11.83s^4 + 329.1s^3 + 5{,}950s^2 + 64{,}160s + 331{,}200} \qquad (3\text{-}58)$$

Using the M method for deriving a circuit for this transfer function, we obtain

$$\frac{E_o}{0.004096s^6 - 0.2705s^5 + 11.83s^4 - 329.1s^3 + 5{,}950s^2 - 64{,}160s + 331{,}200} = \frac{E_i}{0.004096s^6 + 0.2705s^5 + 11.83s^4 + 329.1s^3 + 5{,}950s^2 + 64{,}160s + 331{,}200} = M$$

Hence

$$e_o = 0.004096\overset{6}{M} - 0.2705\overset{5}{M} + 11.83\overset{4}{M} - 329.1\overset{3}{M} + 5{,}950\overset{2}{M} - 64{,}160\overset{1}{M} + 331{,}200M$$

and

$$e_i = 0.004096\overset{6}{M} + 0.2705\overset{5}{M} + 11.83\overset{4}{M} + 329.1\overset{3}{M} + 5{,}950\overset{2}{M} + 64{,}160\overset{1}{M} + 331{,}200M$$

The circuit for realizing these equations is given as Fig. 3-29.

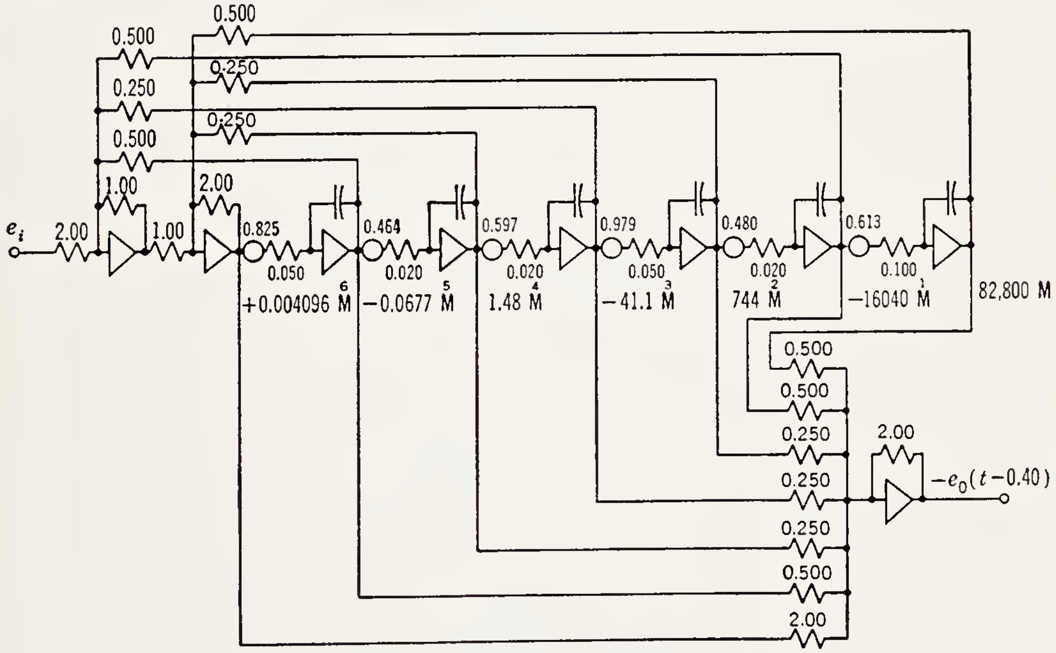

FIG. 3-29. Dead-time circuit for $\tau = 0.40$ sec.

PROBLEMS

3-1. Show that the transfer function for the system described by the block diagram of Fig. P3-1 is given by

$$\frac{E_o}{E_i} = \frac{A_1(s)A_2(s)}{1 + A_2(s)B_2(s) + A_1(s)A_2(s)B_1(s)}$$

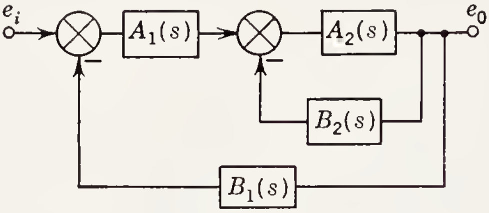

FIG. P3-1.

3-2. Show that the transfer function of the system described by the block diagram of Fig. P3-2 is given by

$$\frac{E_o}{E_i} = \frac{A_1(s)A_2(s) + A_2(s)(E_2/E_i)}{1 + A_1(s)A_2(s)B(s)}$$

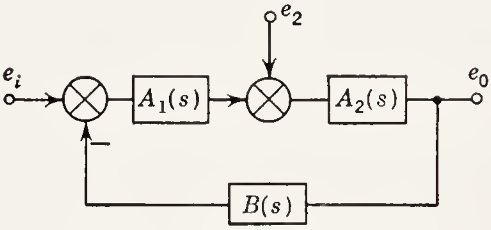

FIG. P3-2.

3-3. Draw Bode plots for the following transfer functions:

(*a*) $$\frac{E_o}{E_i} = \frac{1 + 3s}{(1 + 0.1s)(1 + 2s)}$$

(*b*) $$\frac{E_o}{E_i} = \frac{10}{(1 + 0.1s)(1 + 0.004s + 0.0004s^2)}$$

(*c*) $$\frac{E_o}{E_i} = \frac{2(1 + 0.01s)}{(1 + 0.2s)(4 + 0.4s + s^2)}$$

3-4. Using only one amplifier, set up a circuit to simulate

$$\frac{\theta_o}{\theta_i} = \frac{-800}{s^2 + 20s + 800}$$

3-5. Set up a circuit to simulate the transfer function

$$\frac{E_o}{E_i} = \frac{-1}{1 + 0.05s + 0.0104s^2 + 0.0001s^3}$$

(*a*) by direct solution of the differential equation; (*b*) by network methods (use only one amplifier).

3-6. Show that the transfer function for the amplifier circuit of Fig. P3-6 is given by

$$\frac{E_o}{E_i} = -\frac{[R_2/(R_1 + R_2)]R_3C_1s}{1 + [(R_2 + R_3)/(R_1 + R_2)]R_1C_1s}$$

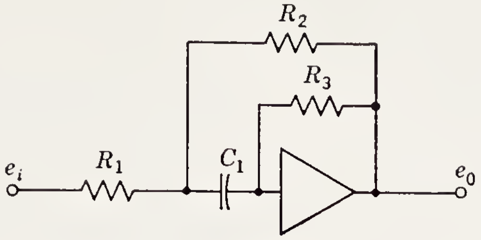

FIG. P3-6.

3-7. Show that the transfer function of the circuit of Fig. P3-7 is given by

$$\frac{E_o}{E_i} = \frac{-1}{1 + (4R_2 + R_1)C_3s + (2R_2C_2 + 2R_1C_1 + R_1C_2)R_2C_3s^2 + R_1R_2^2C_1C_2C_3s^3}$$

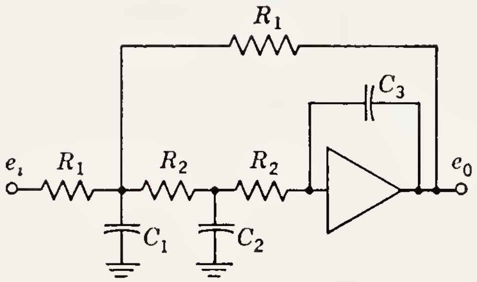

FIG. P3-7.

3-8. Set up a circuit to simulate the transfer function

$$\frac{E_o}{E_i} = \frac{2(1 + 0.5s)}{(1 + s)(1 + 0.2s)}$$

using (*a*) Table 3-1; (*b*) Table 3-2.

3-9. Derive the transfer function for the circuit of Fig. P3-9, using (*a*) nodal analysis; (*b*) the circuits of Table 3-2.

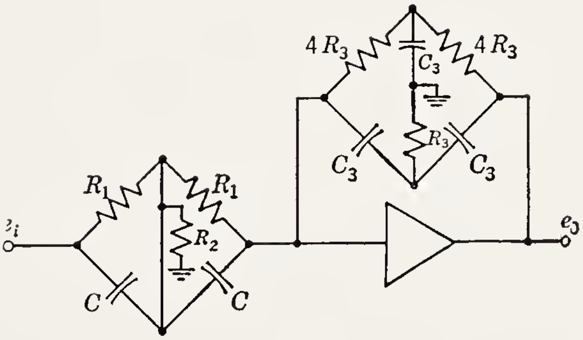

FIG. P3-9.

3-10. Prove that the circuit of Fig. 3-28 simulates the transfer function of Eq. (3-49) if $\tau = 0.1$ sec.

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CHAPTER 4

The Use of Diodes in Analog Computations

4-1. INTRODUCTION

Many of the nonlinear phenomena which the analyst encounters in his study of physical systems may be described in terms of broken straight-line functions. Among these are limit stops, dead space, backlash, or hysteresis, and triangular and square-wave oscillations. These functions can be easily simulated if a switching device is available which is capable of starting and stopping straight-line segments. The simple diode is such a device. We are therefore concerned in this chapter with how to apply the diode to the simulation of these nonlinear phenomena. Chapter 5 gives the application of the relay comparator to the simulation of these same phenomena. In order that we have a clear understanding of what is meant by the term "diode," we shall consider the ideal, or perfect, diode. A perfect diode is a device which possesses the property of unidirectional flow; i.e., it allows the passage of current in only one direction. In particular, a perfect diode consists of two elements, a plate and a cathode, and will conduct if, and only if, its plate voltage is positive with respect to its cathode voltage. It thus serves the function of an electronic switch by stopping the flow of current when the input reaches a certain level. The name originated with that vacuum tube which had only two elements and which possessed approximately this property of unidirectional flow. This property was called rectification, and all devices which had this property were called rectifiers. For this reason we shall define a diode as any device which approximates the ideal performance stated above. It should be understood that, while the circuits presented here are developed for thermionic diodes, other types of diodes could, in most cases, be used equally well.

All diodes are somewhat less than perfect in their performance of the function of unidirectional flow. They operate satisfactorily only over limited voltage and current ranges and generally do not exhibit sharp cutoff characteristics round the point where they stop conducting. They have, however, frequency-response characteristics which in general exceed the requirements for analog computing. For this reason they are becoming more popular than relays (see Chap. 5) for switching operations. Their inherent lack of perfection is generally not critical in most computer operations. The magnitude of the expected errors will be discussed in connection with the circuits that follow. For those cases where diode errors are important, we refer the reader to Sec. 4-10 for a discussion of a technique for idealizing diode performance.

4-2. LIMITING THE SUMMING AMPLIFIER

Suppose we consider the case of a function which varies linearly over certain specified values of its range but exhibits sharp cutoff characteristics when its magnitude reaches some value. Such a function is said to limit, for in this case, if the input continues to increase past the cutoff value, the output (or response) limits at the cutoff value and does not increase any further. Figure 4-1 shows the characteristics of a system which possesses symmetrical positive and negative limits.

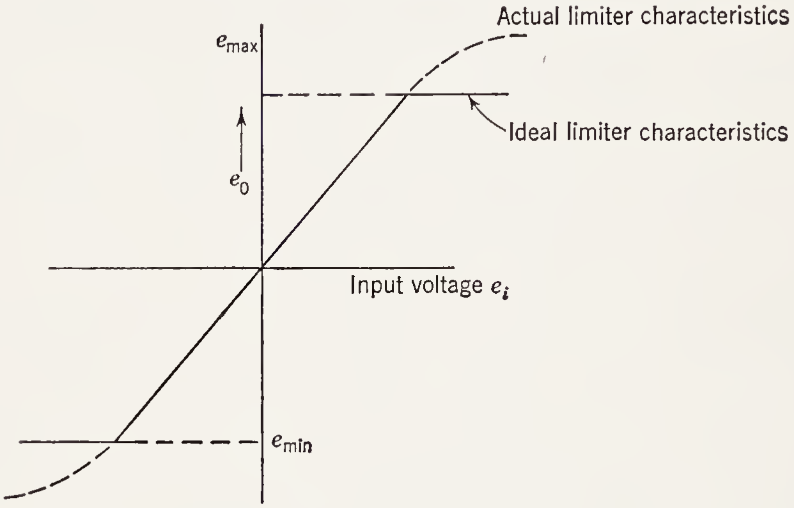


FIG. 4-1. Limiter characteristics.

There are many ways in which diodes can be applied to realize limit characteristics. For an amplifier which is being used as a summer-inverter, there are two possible courses. The input can be limited, thus indirectly limiting the output, or the output can be limited directly. When an amplifier is used as an integrator, it is necessary to limit the output directly.

A simple diode circuit for limiting the input voltage to a summing amplifier consists of an arrangement whereby the input resistor of the amplifier is split into two series resistors. A pair of biased diodes is then inserted from the junction of the split resistors through proper biasing circuits to ground. Such an arrangement is shown in Fig. 4-2 and is known as the shunt input limiter. Since a diode will conduct if, and only if, its plate voltage is positive with respect to its cathode voltage, it can be seen in Fig. 4-2 that diode V_1 will conduct if, and only if, the output voltage e_o of the limiter becomes larger than the bias $(R_F/R_L)e_{\max}$.

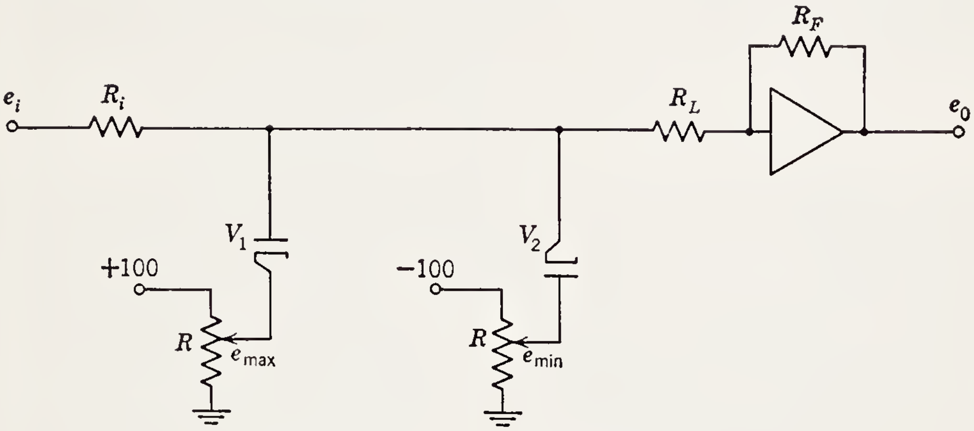


FIG. 4-2. The shunt input limiter.

Similarly, diode V_2 will conduct if, and only if, e_o is less than the bias $(R_F/R_L)e_{\min}$. The bias voltages $e_{\max}$ and $e_{\min}$ may be obtained from standard voltage sources in the computer through potentiometers which permit convenient adjustment of the limiting levels. When the diodes are not conducting, the performance equation of the shunt limiter is

$$e_o = -\frac{R_F}{R_i + R_L} e_i \tag{4-1}$$

When one of the diodes is conducting, the performance equation becomes

$$e_{o_{\text{lim}}} = \frac{R_F}{R_L r + R_i r + R_L R_i} (e_i r \pm 100 a R_i) \tag{4-2}$$

with

$$r = R_D + a(1 - a)R$$

where R_D is the resistance of one diode and a is the bias potentiometer setting. In this equation, if $r <<< R_i$ and $r <<< R_L$, then we obtain

$$e_{o_{\text{lim}}} \doteq \pm 100 \frac{R_F}{R_L} a \tag{4-3}$$

The shunt input limiter will give a limiter characteristic similar to the dashed lines shown on the ideal limiter characteristics of Fig. 4-1. From Eq. (4-2) the final slope of the limiter characteristic is given by

$$\frac{R_F r}{(R_L + R_i)r + R_i R_L} \tag{4-4}$$

Solving Eq. (4-3) for a, there is obtained an approximate value for the bias potentiometer setting; i.e.,

$$a \doteq \frac{R_L}{R_F} \frac{e_{o_{\text{lim}}}}{100} \tag{4-3a}$$

A closer approximation can be obtained by substituting $a' = \dfrac{R_L}{R_F} \dfrac{e_{o_{\text{lim}}}}{100}$ into Eq. (4-5),

$$a = \frac{R_iR_L + a'(R_L + R_i)}{R_FR_i} \frac{e_{o_{\text{lim}}}}{100} \tag{4-5}$$

which is obtained from Eq. (4-2) by letting $r = R_D + a'(1 - a')R$ and then solving for a. However, in practice it is generally easier to calibrate the circuit by trial and error, using Eq. (4-3a) as a starting point. It can be seen from Eqs. (4-2) and (4-4) that, the higher the value of R_F/R_L, the higher the final slope or the poorer the limiter operation. This final slope can be estimated since, in general, the quantity $a(1 - a)R$ is considerably greater than R_D. Therefore a good approximation to the final slope can be had by letting $R_D = 0$ in Eq. (4-2). It is essential to good limiter operation to keep the magnitude of the bias potentiometer

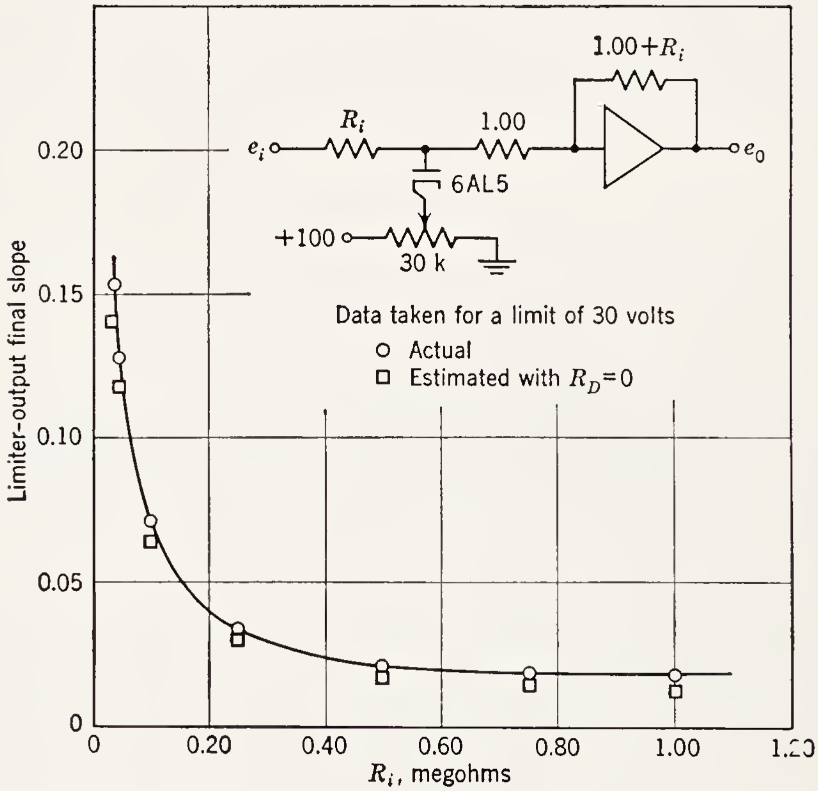


FIG. 4-3. Effect of input resistance on shunt-input-limiter performance.

resistance as small as possible. Figure 4-3 shows actual final slopes for such a limiter (using a 6AL5 diode) and also shows the estimated value of slope obtained by letting $R_D = 0$.

An additional source of error occurs during the transition between the on and off, or conducting and nonconducting, conditions of a diode. During this transition period the diode resistance changes from a very large value to approximately 200 to 10,000 ohms. The resistance of the diode is dependent upon the current flowing through the diode, as shown in Fig. 4-4 for a typical 6AL5. Since the current through the diode is approximately proportional to $1/R_i$, small values of R_i yield small values of diode resistance. However, the smaller R_i, the more effect the diode resistance has on the slope of the final output. The result is that round-off error can be made small only at the expense of poor final slope. In practice it is generally better to accept greater round-off error for smaller slope. Another factor to be considered is the ratio of R_i/R_L. For constant r and constant R_f the final slope will be a minimum when $R_i = R_L$ since the quantity

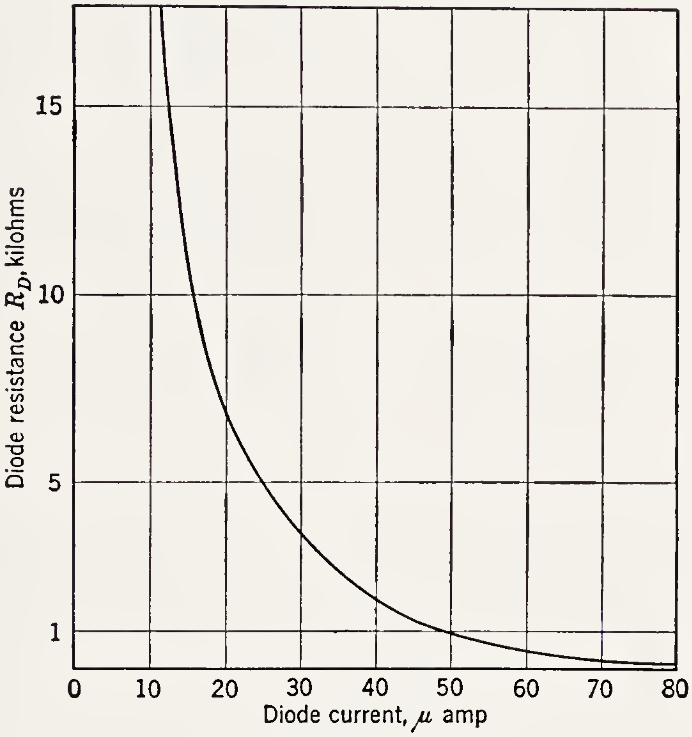


FIG. 4-4. Variation of diode resistance with diode current for a typical 6AL5.

$$R_L R_i + r(R_L + R_i) = R_L R_i + rkR_f = R_L R_i + K$$

will be a maximum. As mentioned above, R_D and therefore r will increase as R_i increases, but, for $R_i \leq 2$ megohms, this effect is small.

Summarizing, the shunt input limiter yields best accuracy when:

1. The bias potentiometer resistance is small.
2. $R_i = R_L = 0.5$ megohm or greater.
3. $R_F/(R_i + R_L) \leq 1$.

It is also possible to use the shunt limiter in the feedback of an operational amplifier, in which case we call it a shunt output limiter. The limiting takes place when the diodes do not conduct; therefore R_i and R_L must be small. When either diode conducts, the gain is increased. It may be necessary to place a resistance in series with the diode to achieve the desired gain. The current through the diode must also be kept within the limits of the diode specification. This circuit is less flexible and less accurate than other circuits which produce equivalent results and as a consequence is seldom, if ever, used for limiting operations. It does, however, lend itself better to function-generation techniques and will be discussed further in Sec. 4-7.

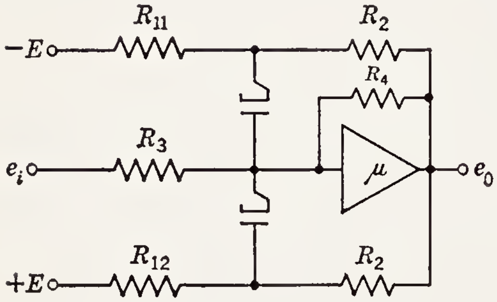


FIG. 4-5. Series output limiter.

Performance similar to the shunt input limiter can be obtained by the use of the series output limiter shown in Fig. 4-5. The greatest disadvantage of the series output limiter arises from the fact that certain of the computing amplifiers in use today are not stable with feedback resistors less than 0.10 megohm. With amplifiers that are stable with small feedbacks, however, it has the advantages over the shunt limiter of a higher switching speed and greater operational flexibility. For example, the series output limiter may have an extremely high gain for very small signals and almost zero incremental gain for large signals; as was pointed out previously, this is not feasible with the shunt input limiter.

The circuit-performance equation for the series output limiter is given by the nodal equation

$$\left(\frac{e_o}{\mu} - e_i\right)\frac{1}{R_3} + \left(\frac{e_o}{\mu} - e_o\right)\frac{1}{R_4} = 0 \tag{4-6}$$

as long as the diodes do not conduct. When either diode conducts, the nodal equation becomes

$$\left(\frac{e_o}{\mu} - e_i\right)\frac{1}{R_3} + \left(\frac{e_o}{\mu} - e_o\right)\frac{1}{R_4} + \left(\frac{e_o}{\mu} - e_o\right)\frac{1}{R_2} + \left(\frac{e_o}{\mu} \pm E\right)\frac{1}{R_{1i}} = 0 \qquad i = 1, 2 \tag{4-7}$$

For large absolute values of the amplifier gain μ, it will be permissible to neglect e_o/μ in the above equations. Then the transfer characteristics are simply

$$e_o = -\frac{R_4}{R_3} e_i \tag{4-8}$$

if the diodes are not conducting and

$$e_o = \pm \frac{R_2 R_4}{(R_2 + R_4) R_{1i}} E - \frac{R_2 R_4}{(R_2 + R_4) R_3} e_i \tag{4-9}$$

if one of the diodes is conducting. In Eq. (4-9) the quantity

$$\frac{R_2 R_4}{(R_2 + R_4) R_3}$$

is the final slope of the limiter characteristic. If we make $R_2 \ll R_4$ then the final slope is given by approximately R_2/R_3. If in addition $R_2 \ll R_3$, then the final slope will have a small value. This fact is illustrated in the example to follow.

The procedure for using the series output limiter is to set the limit by applying reference voltages $\pm E$ and selecting appropriate resistors R_{1i} and R_2 to conform to

$$e_{o_{\text{lim}}} = \pm \frac{R_2 R_4}{(R_2 + R_4) R_{1i}} E \tag{4-10}$$

After the limit is reached, the output will remain more nearly constant if R_2 is small compared with R_3. The limit is then given approximately by

$$e_{o_{\text{lim}}} = \pm \frac{R_2}{R_{1i}} E \tag{4-11}$$

For the case of symmetrical limits $R_{11} = R_{12}$ and will be simply designated R_1.

To illustrate the application of the series output limiter, let us consider the following example.

Example 1. It is desired to limit an amplifier so that the absolute value of its output will not exceed 40 volts; within this range the amplifier is to have unity gain.

Case 1. The following values for the circuit constants are selected:

$R_3 = R_4 = 1$ megohm
$R_2 = 0.02$ megohm
$E = 100$ volts
$R_1 = 0.05$ megohm

Then by Eq. (4-9)

$$e_o = \pm 39.2 - 0.0196 e_i$$

after the limit has been reached.

Case 2. The following values for the circuit constants are selected:

$R_3 = R_4 = 10$ megohms
$R_2 = 0.02$ megohm
$E = 100$ volts

Then by Eq. (4-9)

$$e_o = \pm 39.9 - 0.001997 e_i$$

after the limit has been reached. Figure 4-6 illustrates the difference obtained using the two different sets of circuit values.

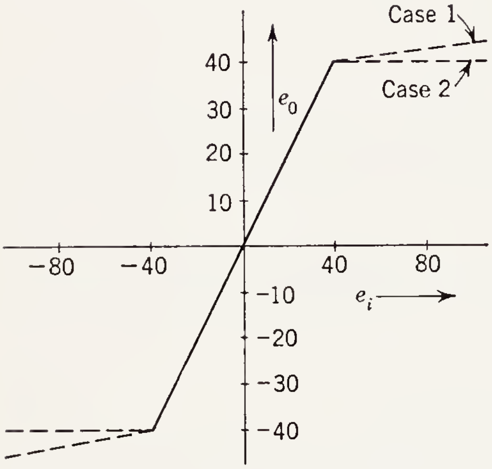


FIG. 4-6. Series-output-limiter performance.

A little used, but extremely effective, circuit for limiting the output of a summing amplifier is the series input limiter. This limiter, shown in Fig. 4-7, differs from the two described previously in that the diodes conduct only between the limits. Since they stop conducting when the limit is reached, the limit is not so dependent on diode characteristics. Once the transition from conducting to nonconducting is completed, the limit is constant within the accuracy of the computer. There are, however, two minor disadvantages to this limiter. First, the circuit relationships are somewhat complicated. Second, since the diodes are conducting during the period of linear operation, the diode characteristics may cause some error in the operation. With the diodes in use in the analog computers, this latter problem is usually negligible, although, if the two diodes used are not matched, errors of 0.10 to 0.20 volt may be expected within the region of linear operation. These problems can be minimized by adjusting the gain and bias to give optimum results. The circuit rela-

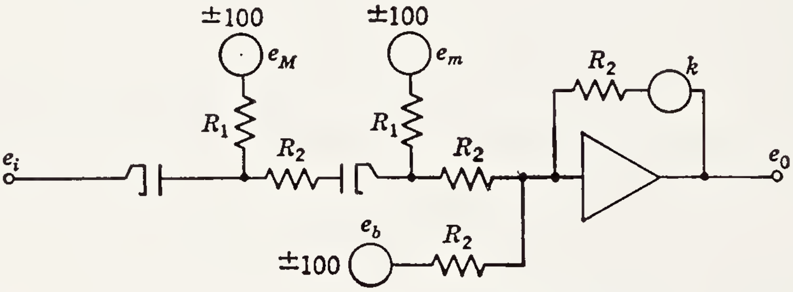


FIG. 4-7. The series input limiter.

tionships can be summarized in the following manner: From the circuit of Fig. 4-7 if we let

$$R_1 = bR_2$$

and

$$e_b = \frac{-e_m}{1 + 2b}$$

we have

$$e_o = \begin{cases} -\dfrac{1}{k}\dfrac{be_m}{(1+b)(1+2b)} & \text{for} \quad e_i \leq \dfrac{e_m}{1+b} \\ -\dfrac{1}{k}\dfrac{be_i}{1+2b} & \text{for} \ \dfrac{e_m}{1+b} \leq e_i \leq \dfrac{(1+2b)e_M + be_m}{b^2+3b+1} \\ -\dfrac{1}{k}\dfrac{be_M + \dfrac{b^2}{1+2b}e_m}{b^2+3b+1} & \text{for} \quad e_i \geq \dfrac{(1+2b)e_M + be_m}{b^2+3b+1} \end{cases} \tag{4-12}$$

Thus if, $k = b/(1 + 2b)$, the summer has a one-to-one gain during its linear operation, and the relationships of Eq. (4-12) become

$$e_o = \begin{cases} \dfrac{-e_m}{1+b} & \text{for} \quad e_i \leq \dfrac{e_m}{1+b} \\ -e_i & \text{for} \ \dfrac{e_m}{1+b} \leq e_i \leq \dfrac{(1+2b)e_M + be_m}{b^2+3b+1} \\ -\dfrac{(1+2b)e_M + be_m}{b^2+3b+1} & \text{for} \quad e_i \geq \dfrac{(1+2b)e_M + be_m}{b^2+3b+1} \end{cases} \tag{4-13}$$

Two examples which follow will demonstrate the range in limits which can be set and the practical use of the circuit.

Example 2. The output of a summing amplifier will be limited to operate between the values +50 volts and −40 volts. Within these limits the amplifier will have unity gain. Let $R_1 = R_2 = 1.0$ megohm. Then $b = 1$, and we have from Eq. (4-13)

$$e_o = \begin{cases} -\dfrac{e_m}{2} & \text{for } e_i \leq \dfrac{e_m}{2} \\ -e_i & \text{for } \dfrac{e_m}{2} \leq e_i \leq \dfrac{3e_M + e_m}{5} \\ -\dfrac{3e_M + e_m}{5} & \text{for } e_i \geq \dfrac{3e_M + e_m}{5} \end{cases}$$

Since the upper limit is +50 volts, we have $e_m = -2(50) = -100$ volts. Using this result and the desired lower limit of −40 volts, we get

$$-40 = -\frac{3e_M - 100}{5}$$

and so $e_M = +100$ volts. Since $e_b = -e_m/(1 + 2b)$, we have

$$e_b = 100/3 = 33.3 \text{ volts}$$

Since $k = b/(1 + 2b)$ for unity gain, we have $k = 1/3 = 0.333$. Using these values, the circuit of Fig. 4-8 results.

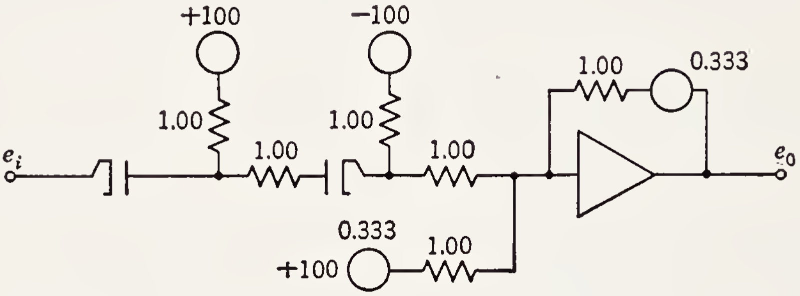


FIG. 4-8. Example 1, series input limiter.

Example 3. It is desired to limit the output of a summer to values between +10 volts and +90 volts. The summer is to have unity gain between these values. Let $R_1 = 0.1$ megohm and $R_2 = 1.0$ megohm. Then $b = 0.1$, and we have for the upper limit

$$90 = \frac{-e_m}{1.10}$$

and so

$$e_m = -99 \text{ volts}$$

For the lower limit we have

$$10 = \frac{-(1.2e_M - 9.9)}{1.31}$$

and so

$$e_M = -2.67 \text{ volts}$$

Now

$$k = \frac{0.1}{1.2} = 0.085$$

and

$$e_b = \frac{99}{1.2} = 82.50$$

giving us the circuit shown in Fig. 4-9.

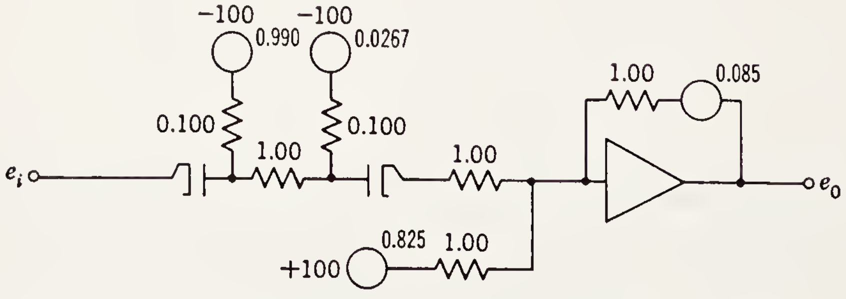


FIG. 4-9. Example 2, series input limiter.

The series input limiter also has the advantage of being suitable for use with idealized diodes.* This combination yields the most precise limit of all the circuits given and does not have a detrimental effect on the linear characteristics.

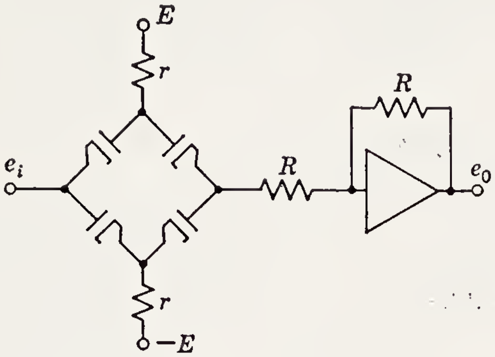


FIG. 4-10. Bridge limiter circuit.

Perhaps the most common limiter not yet discussed is the bridge input limiter shown in Fig. 4-10. The following relationships exist for this circuit:

$$e_o \doteq \begin{cases} +\dfrac{R}{R+r}E & e_i \leq -\dfrac{R}{R+r}E \\ -e_i & -\dfrac{R}{R+r}E \leq e_i \leq \dfrac{R}{R+r}E \\ -\dfrac{R}{R+r}E & e_i \geq \dfrac{R}{R+r}E \end{cases} \tag{4-14}$$

This circuit gives an excellent hard limit for symmetrical limit values and is very simple to set up. There are, however, two disadvantages to this limiter. First, it has a small output bias if the limits are not symmetrical. Second, it is not possible to limit between voltages of the same sign.

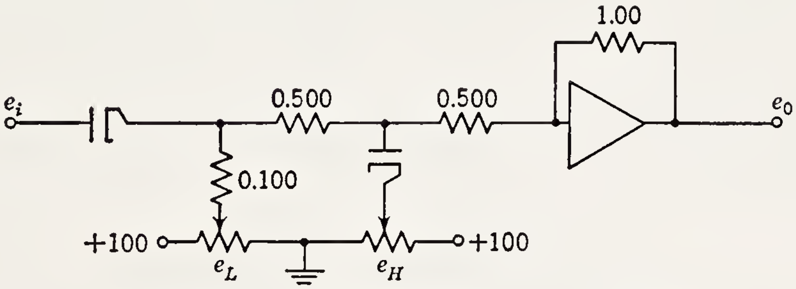


FIG. 4-11. Combined shunt-series input limiter.

Figure 4-11 shows a combination shunt and series input limiter which combines some of the features of each. It has a sharp limit on one side, a fair limit on the other. In addition, it is considerably easier to set up than the series input limiter. The output can be summarized as

---

* See Sec. 4-10.

$$e_o = \begin{cases} -\dfrac{1.0}{1.1} e_L & \text{for} \quad e_i \leq \dfrac{e_L}{1.1} \\ -e_i & \text{for} \dfrac{e_L}{1.1} \leq e_1 \leq 2e_H \\ -2e_H & \text{for} \quad e_i \geq 2e_H \end{cases} \tag{4-15}$$

4-3. LIMITING AN INTEGRATOR

A number of circuits exist for limiting an integrator, including some involving cutoff relays. One of the simplest uses a series output limiter, as shown in Fig. 4-12. The performance of the series output limiter when used to limit an integrating amplifier is given by Eqs. (4-8) and (4-9) with $1/Cs$ substituted for R_4. When either diode conducts, the amplifier no longer acts as an integrator but becomes a first-order lag circuit with very low gain and time constant. If $R_3 = 1$ megohm and $R_2 = 10$ kilohms, the maximum error of the limit will be of the magnitude of 1 volt.

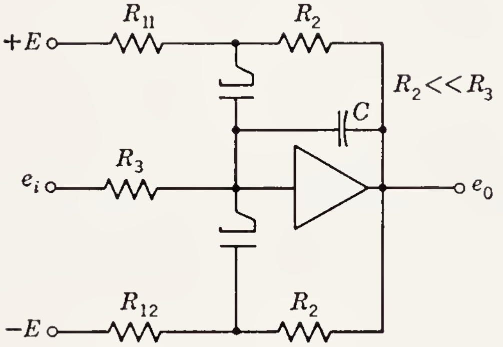


FIG. 4-12. Limiting an integrator with a series output limiter.

The residual slope remaining after the limit value is reached can be reduced by inserting a summing amplifier after the limited integrator and using a shunt, series, or bridge limiter to limit the summer. Extreme accuracy can be obtained with any of these limiters, since the voltage input only slightly exceeds the desired limits. The following example illustrates how an exact limit can be achieved by using a series output limiter on an integrating amplifier, followed by a shunt-input-limited summing amplifier.

Example. It is desired to limit the output of an integrator between +20 volts and −40 volts. From Eq. (4-9) we have, by replacing $1/R_4$ with Cs,

$$e_o = \frac{-R_2 e_i}{R_3(1 + R_2Cs)} \mp \frac{R_2E}{R_i(1 + R_2Cs)} \tag{4-16}$$

Let $R_2 = 0.01$ megohm, $R_3 = 1.0$ megohm, $C = 1.0\ \mu\text{f}$, $E = 100$ volts, and we have

$$R_{11} = -\frac{(0.01)(100)}{-40} = 0.025 \text{ megohm}$$

and

$$R_{12} = \frac{(0.01)(100)}{20} = 0.050 \text{ megohm}$$

For the shunt limiter let $R_i = R_L = 1.0$ megohm, and let $R = 30$ kilohms. Then from Eq. (4-5) the bias potentiometer settings are approximately 0.202 and 0.1005. Figure 4-13 shows the resulting circuit.

When limiting an integrator, one is frequently tempted to insert a summing amplifier behind the integrator and limit its value to the integrator limit values, leaving the integrator unlimited. This procedure is not valid! For if the integrator keeps integrating after its desired limit has been reached, then, when the sign of the input to the integrator changes, the output of the summer does not immediately decrease, as it should, but must wait until the output of the integrator falls below the limit value. This yields an incorrect result.

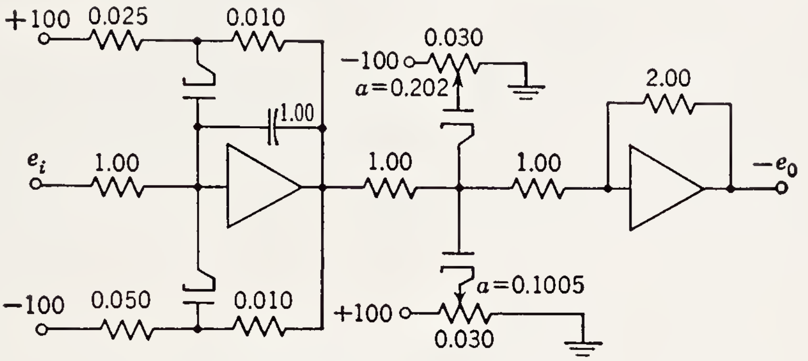


FIG. 4-13. Example showing a sharp limiter for an integrator.

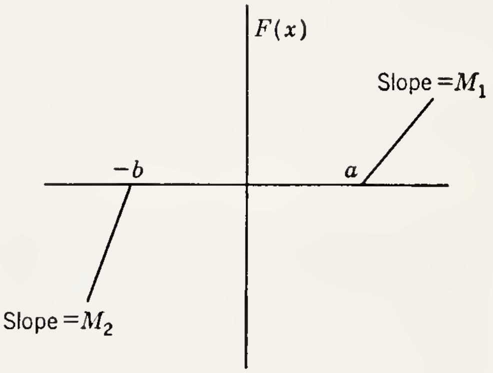


FIG. 4-14. Dead-space characteristic.

4-4. SIMULATION OF DEAD SPACE

Dead space is encountered in many mechanical and electromechanical transducers such as tachometers, accelerometers, gyroscopes, and pressure gages. The phenomenon is characterized by a region of no sensitivity, i.e., an inert zone, or dead space. Mathematically it can be defined by the relationship

$$F(x) = \begin{cases} m_1(x - a) & x \geq a \\ 0 & -b \leq x \leq a \\ m_2(x + b) & x \leq -b \end{cases} \qquad (4\text{-}17)$$

A graphical representation is given in Fig. 4-14. There are many ways to simulate dead space using diodes. Presented in Fig. 4-15 is a circuit to simulate dead space as described above. In Fig. 4-16 are two circuits to simulate the special case for which $M_1 = M_2$.

The following relationships describe the operation of the circuit of Fig. 4-15:

$$e_o = \begin{cases} -a_2 \dfrac{R_5}{R_2} e_i & e_i \le -\dfrac{a_1 R_2}{a_2 R_1} 100 \\ 0 & -\dfrac{a_1 R_2}{a_2 R_1} 100 \le e_i \le \dfrac{a_4 R_3}{a_3 R_4} 100 \\ -a_3 \dfrac{R_5}{R_3} e_i & e_i \ge \dfrac{a_4 R_3}{a_3 R_4} 100 \end{cases} \qquad (4\text{-}18)$$

The circuit of Fig. 4-15 is the most general circuit for simulating dead space. The dead-space region can be centered round any part of the operating range, M_1 does not have to equal M_2, and the dead space can be biased up or down to have any desired value.

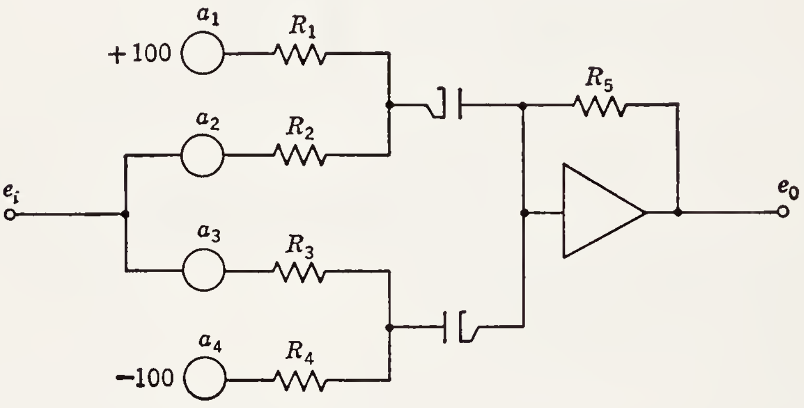


FIG. 4-15. A circuit for dead space.

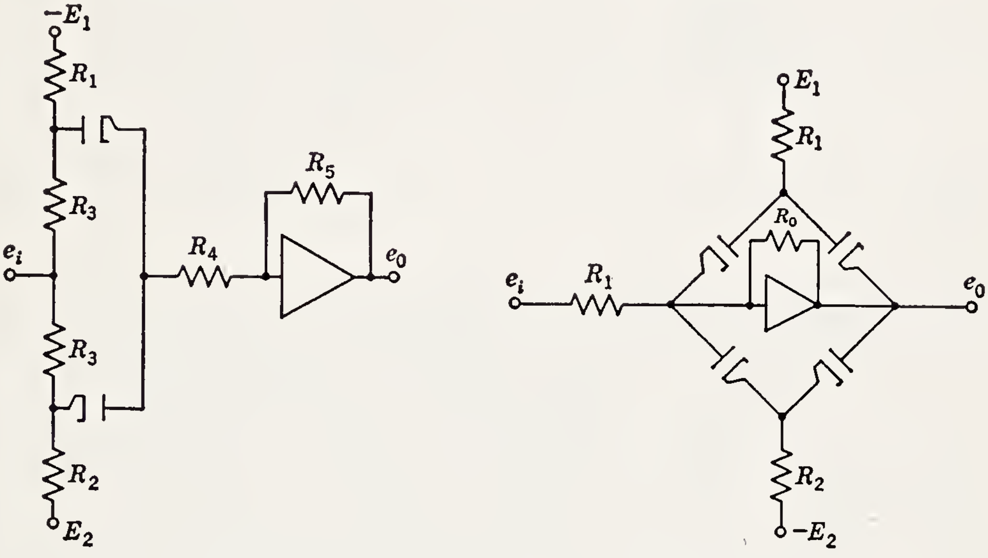
The dead-space circuit of Fig. 4-16*a* yields good results although it is subject to the usual errors of diode round-off. The gains and slopes can be changed independently, but the bias must be changed when this is done. The circuit of Fig. 4-16*a* can also be used for general dead space, but it is better adapted to the special case $M_1 = M_2 = M$. The output of this circuit is governed by the relationships

$$e_o = \begin{cases} -\dfrac{R_5}{R_3 + R_4 + R_3 R_4 / R_2}\left(e_i + \dfrac{R_3}{R_2} E_2\right) & e_i \le -\dfrac{R_3}{R_2} E_2 \\ 0 & -\dfrac{R_3}{R_2} E_2 \le e_i \le \dfrac{R_3}{R_1} E_1 \\ -\dfrac{R_5}{R_3 + R_4 + R_3 R_4 / R_1}\left(e_i - \dfrac{R_3}{R_1} E_1\right) & e_i \ge \dfrac{R_3}{R_1} E_1 \end{cases} \qquad (4\text{-}19)$$

This circuit is similar to the previous circuit and gives essentially the same type of results. The circuit of Fig. 4-16*b* has the following output:

$$e_o = \begin{cases} -\dfrac{R_0}{R_1}(e_i + E_2) & e_i \leq -E_2 \\ 0 & -E_2 \leq e_i \leq E_1 \\ -\dfrac{R_0}{R_1}(e_i - E_1) & e_i \geq E_1 \end{cases} \qquad (4\text{-}20)$$

This circuit is simple to adjust but yields poor results in the dead space. With 6AL5s the error of the dead space may be as high as ±0.6 volt.



(*a*) (*b*)

FIG. 4-16. Circuit for simulation of dead space when $M_1 = M_2$.

If the accuracy of these circuits is not sufficient, an excellent circuit utilizing idealized diodes is given in Sec. 4-10.

4-5. HYSTERESIS AND BACKLASH

According to the "International Dictionary of Physics and Electronics"* hysteresis is "in general, the phenomenon exhibited by a system whose state depends on its previous history." A common form of hysteresis is the behavior of a magnetic material in a magnetic field, which has a characteristic of the form shown in Fig. 4-17. Somewhat similar behavior is exhibited by dielectrics, while some solids exhibit a similar relationship between stress and strain. Backlash is a specific type of hysteresis which is common in mechanical systems and, in particular, in gear trains. It is characterized by the behavior shown in Fig. 4-18.

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* "The International Dictionary of Physics and Electronics," p. 434, D. Van Nostrand Company, Inc., Princeton, N.J., 1956.

As an example of a case where backlash occurs, consider the system consisting of the two gears of Fig. 4-19, where gear 1 is the driver and gear 2 is not spring-loaded. It can be easily seen that if θ_i were to reverse direction the teeth of gear 1 must pass through a distance δ_r, that is, rotate through an angle δ, before they again engage gear 2. This results in the characteristic shown in Fig. 4-18.

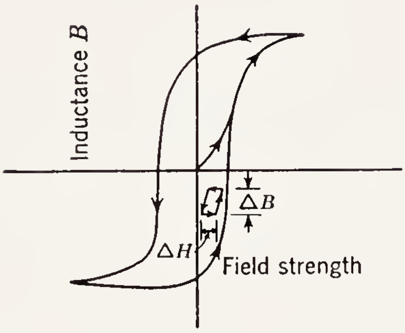


FIG. 4-17. Magnetic-hysteresis characteristic.

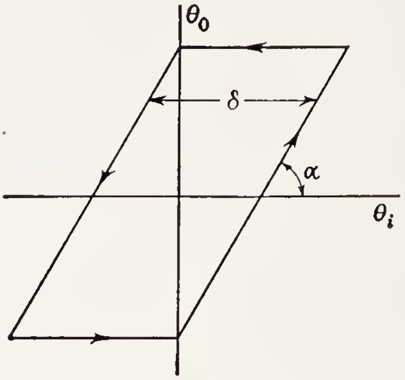


FIG. 4-18. Backlash characteristic.

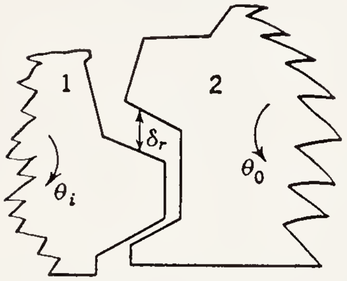


FIG. 4-19. A system with backlash.

The simulation of hysteresis on an analog computer is limited to certain specific cases. The type that is most commonly simulated is backlash. There are many circuits available for the generation of backlash, all of which utilize the same general technique. The simulation is achieved by combining one of the dead-space techniques of Sec. 4-3 with a capacitive network which provides a memory. A simple circuit which achieves this end is given in Fig. 4-20. The output of this circuit has the following constants:

$$\tan\alpha = \frac{C_1/C_2}{1 + R_2/R_1 + (R_D + R_2 + R_DR_2/R_1)C_1s} \qquad \delta = \frac{200R_2}{R_1} \tag{4-21}$$

where R_D is the resistance of the diode. From Eq. (4-21) it is apparent that this circuit has a time lag associated with it, but if R_2 is made small, this lag will be negligible in most cases. While this circuit is very simple, it has the disadvantage of being an open-loop calculation and therefore subject to capacitor drift. The drift of this circuit is generally no worse than that of an integrator; so in most problems it imposes a no more stringent time limit than is usual. The effect of the time lag can be partially canceled by placing a resistor in the feedback in series with the capacitance, as shown in Fig. 4-21. In this case the circuit has the

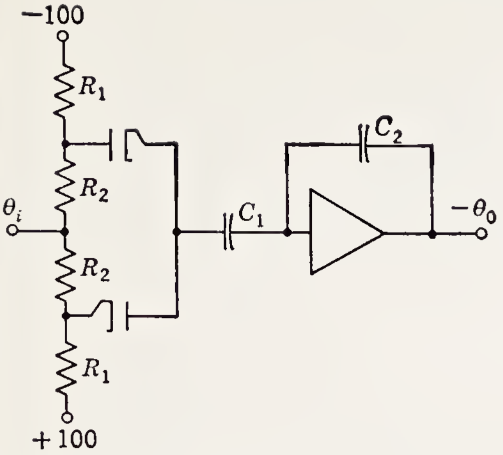


Fig. 4-20. A one-amplifier hysteresis circuit.

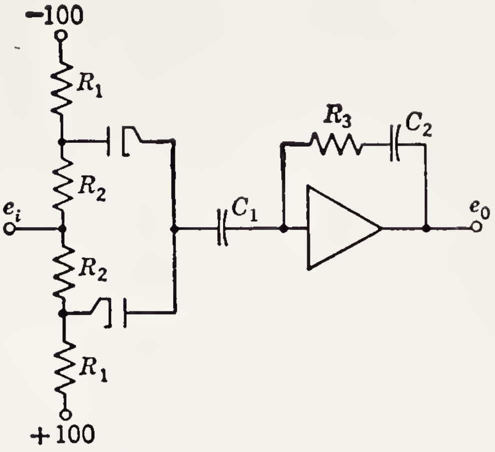


Fig. 4-21. Alternative one-amplifier backlash circuit.

relationships

$$\delta = \frac{200R_2}{R_1}$$

$$\tan\alpha = \frac{(C_1/C_2)(1 + R_3C_2s)}{1 + R_2/R_1 + (R_D + R_2 + R_DR_2/R_1)C_1s}$$

If now the input is of a high frequency, so that the diode switching occurs rapidly, it is possible to cancel the effect of the lag. Thus if

$$R_3C_2 = R_2C_1 \doteq \left(R_D + R_2 + \frac{R_DR_2}{R_1}\right)C_1s \tag{4-22}$$

for $R_D \to 0$, the lag is canceled except for the transition period of the diode. For a sinusoidal input of 5 cps the output of this circuit will be approximately that of Fig. 4-22. The spike decreases in duration as the input frequency increases. This circuit enables one to simulate backlash in a high-frequency system provided that the frequency does not vary over too wide a range.

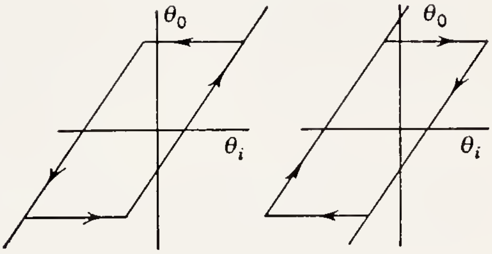


Fig. 4-22. Output of alternative one-amplifier backlash circuits for a 5-cps sinusoidal input.

Figure 4-23 shows a closed-loop circuit for simulating the backlash characteristics. This circuit is governed by the relationships

$$\delta = \frac{200R_2}{R_1}$$

$$\tan\alpha = \frac{K}{1 + \dfrac{R_5C_1(R_2 + R_3 + R_2R_3/R_1)}{R_4}s} \tag{4-23}$$

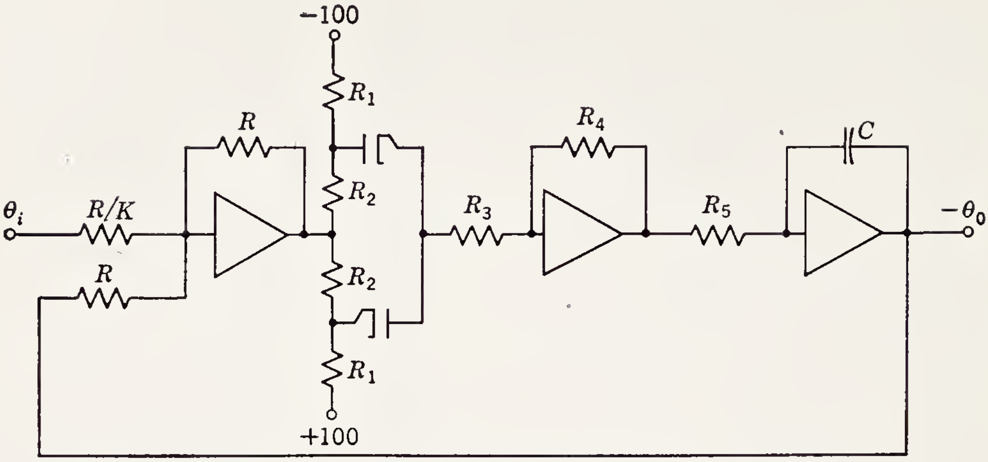


FIG. 4-23. A closed-loop backlash circuit.

This circuit gives similar results to that of Fig. 4-20, with one exception: the output cannot drift except in the dead zone. It is, however, more subject to drift in the dead band.

4-6. ABSOLUTE VALUE

The problem of obtaining absolute value occurs often in practical problems. The mathematical notation for the absolute value of x is $|x|$, and it is characterized by the equation

$$|x| = \begin{cases} +x & x \geq 0 \\ -x & x \leq 0 \end{cases} \tag{4-24}$$

This characteristic is shown graphically in Fig. 4-24.

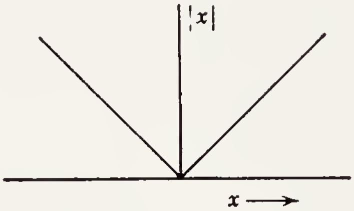
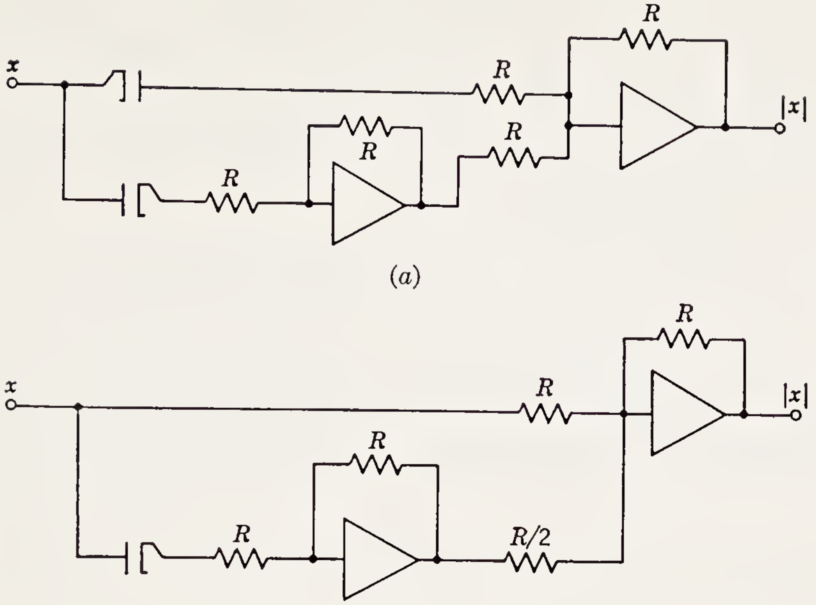


FIG. 4-24. Absolute-value characteristic.

Figure 4-25 shows two of the many possible diode circuits for generating the absolute value. In general, the circuit of Fig. 4-25*b* is more desirable since it uses only one diode. Another type of circuit uses the diode or diodes as a shunt. This is done by splitting the input resistor and connecting the diode from the mid-point to ground. In general, however, the series diode networks shown yield better results.



(a)

(b)

FIG. 4-25. Absolute-value circuits.

4-7. THE COMPARATOR AND ITS APPLICATIONS

The attempt to simulate dry, or coulomb, friction, the characteristic of which can be described mathematically by the function

$$F(x) = \begin{cases} a & x < 0 \\ 0 & x = 0 \\ -a & x > 0 \end{cases} \tag{4-25}$$

leads us to the study of the comparator. The comparator can be described by the function

$$F(x) = \begin{cases} A & x < C \\ 0 & x = C \\ -B & x > C \end{cases} \tag{4-26}$$

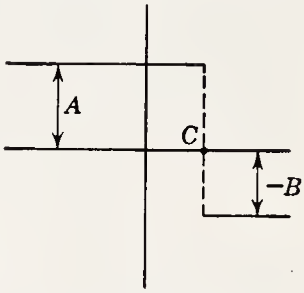


FIG. 4-26. Ideal comparator characteristics.

As can be seen by inspection, Eq. (4-25) is but a special case of Eq. (4-26), so that dry friction can be simulated by the comparator. This function is shown graphically as Fig. 4-26. In practice the comparator characteristics are approximated by the function

$$F_1(x) = \begin{cases} A - \delta\left(x - C + \dfrac{A}{\mu}\right) & x \le C - \dfrac{A}{\mu} \\ \mu(x - C) & C - \dfrac{A}{\mu} \le x \le C + \dfrac{B}{\mu} \\ -B - \delta\left(x - C - \dfrac{B}{\mu}\right) & x \ge C + \dfrac{B}{\mu} \end{cases} \tag{4-27}$$

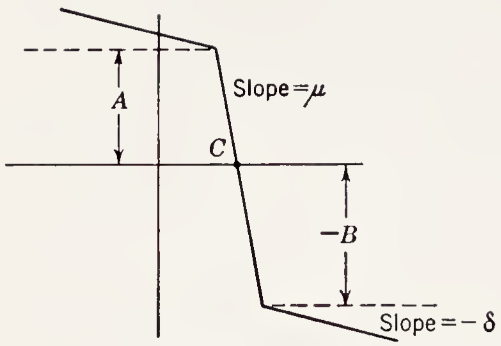


FIG. 4-27. Comparator characteristics realized.

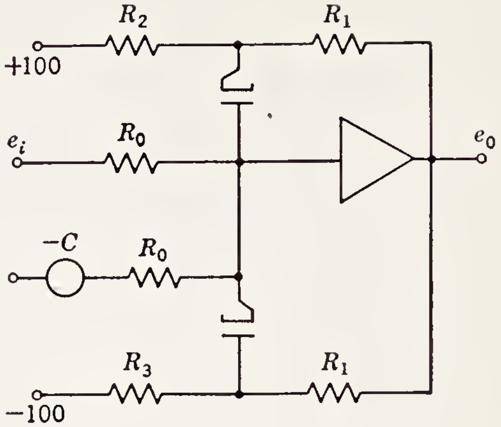


FIG. 4-28. Comparator circuit.

where $\mu \gg 0$. This function is illustrated in Fig. 4-27. A circuit which approximately simulates this function is given in Fig. 4-28. In this circuit

$$A = \frac{R_1}{R_2} 100 \qquad \text{and} \qquad B = \frac{R_1}{R_3} 100$$

For best results R_1 should be much less than R_0. It will be seen that this circuit is simply a high-gain amplifier which is series-output-limited. As soon as the sum of the voltages going into the amplifier changes sign, the output e_o moves rapidly to the approximate limit value, with the slope being given by the gain of the amplifier. For this circuit $\delta = R_1/R_0$.

The final slope δ can be made very close to zero by using a second limiter following the circuit of Fig. 4-28. Since the usual amplifier gain is of the order of magnitude of 10^8, and since δ can be made to approach zero, this circuit can be used to give results very close to the theoretical comparator.

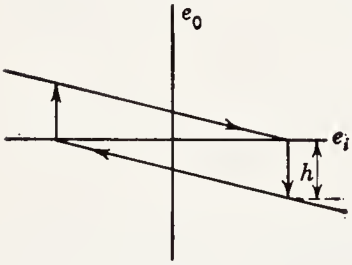


FIG. 4-29. The bistable multivibrator output.

The comparator finds many uses besides the previously mentioned simulation of dry friction. It is excellently adapted to a variety of switching circuits. The following examples will serve to illustrate a few of the possible uses.

Example 1. The Bistable Multivibrator. The output of the bistable multivibrator is shown in Fig. 4-29. This device is used in many switching circuits and can also be used to simulate the switching characteristics of a relay. A circuit which performs this function is given in Fig. 4-30. For this circuit the height h of the jump is given by the equation $h = 2/R$.

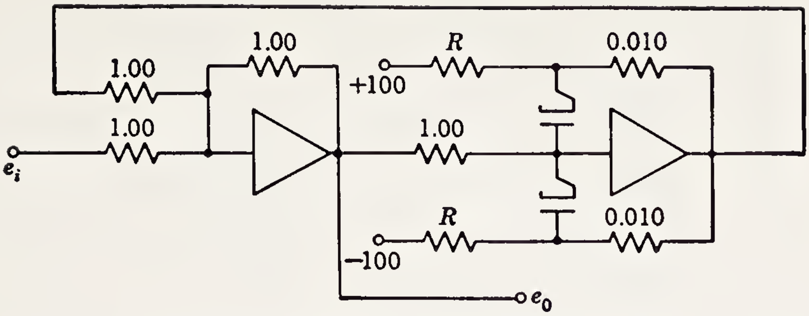
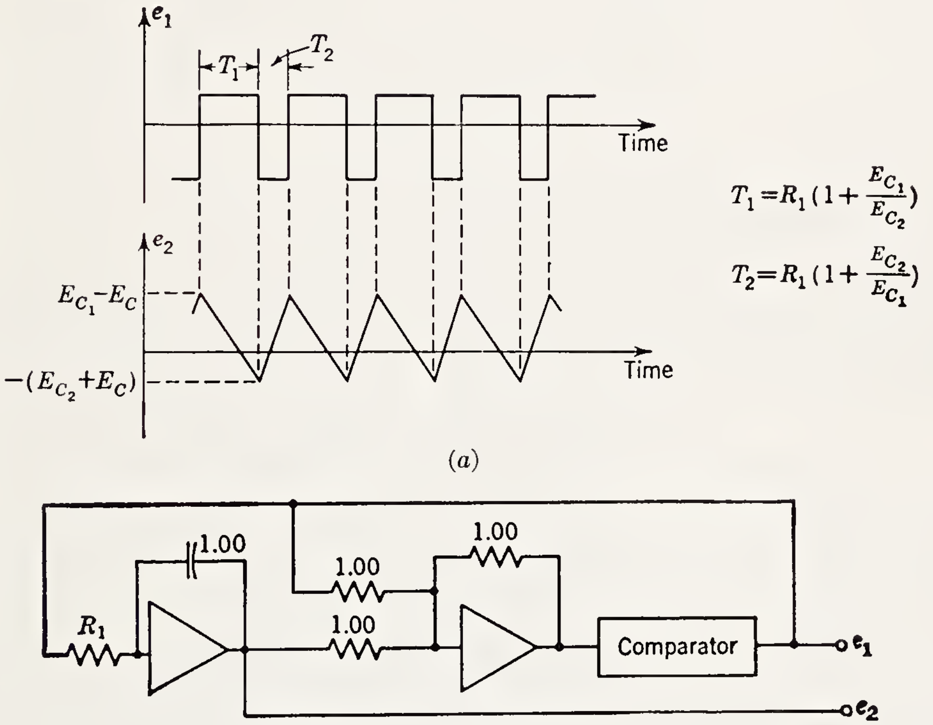


FIG. 4-30. A bistable multivibrator circuit.

Example 2. The Free-running Multivibrator. The free-running multivibrator can be used to generate the square-wave and triangular-wave outputs shown in Fig. 4-31*a*. The circuit for generating these functions is given in Fig. 4-31*b*.



(*a*)

(*b*)

FIG. 4-31. The free-running multivibrator.

Example 3. An Electronic Switch. The comparator can be used to provide for the switching of one variable as a function of the relationship of two other variables. The circuit of Fig. 4-32 shows how this can be done and also shows the output relationships and the function limitations. It can be seen that, by replacing X or Y by an appropriate constant, the switching can be done as a function of one variable.

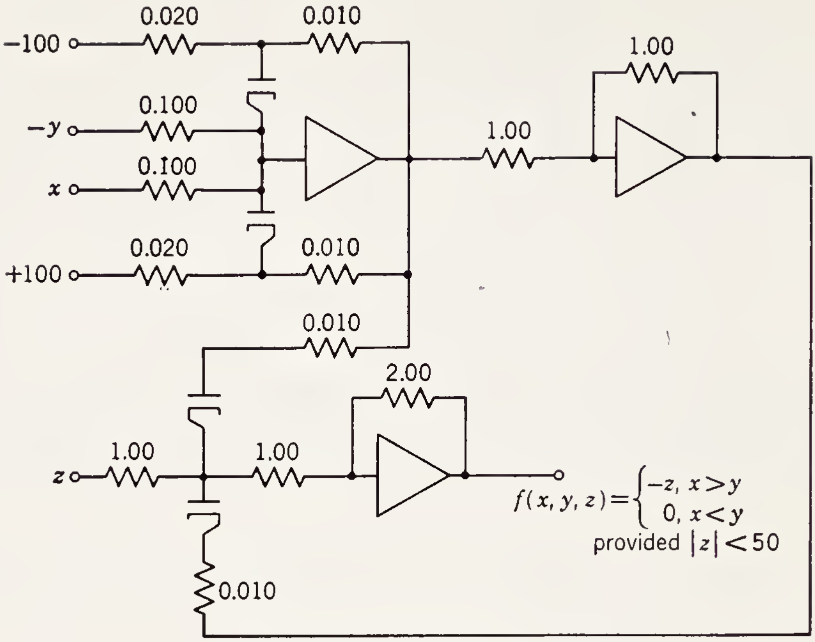


FIG. 4-32. A diode switch.

4-8. THE GENERATION OF SIMPLE ARBITRARY FUNCTIONS

By slightly changing the basic limiter circuits, a great variety of simple monotonic functions can be generated. This is done by approximating the function to be generated by a series of straight-line segments as illustrated in Fig. 4-33. Series, shunt, or combination series-shunt circuits may be used. Each has certain advantages and limitations which will be pointed out.

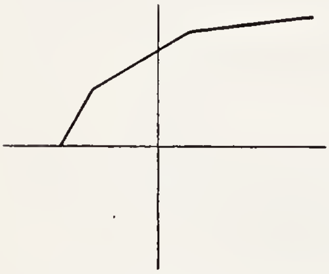


FIG. 4-33. Approximation of an arbitrary function with straight-line segments.

In general, it is not desirable to have sharp corners on the function; so diode round-off is now an advantage instead of a disadvantage. For this reason the shunt limiter, which had rather poor limiting qualities, provides one of the best circuits for the generation of arbitrary functions. There is one other consideration that often determines the best type of circuit to use. This is the consideration of accuracy about zero input. In general, best accuracy can be obtained about zero if the circuit is linear in that region, i.e., if no diodes are conducting. If this is true, then accuracy about zero is solely a function of the precision resistors used.

The examples which follow will show how a simple monotonic function can be generated using a number of possible circuitries. Because of the simplicity of function these circuits can be built up easily on the patch board. The generation of complex functions will be discussed in detail in Chap. 7.

Consider the function

$$F(x) = \begin{cases} -\frac{9}{5}x - 26 & x \leq -20 \\ -\frac{1}{2}x & -20 \leq x \leq 40 \\ -\frac{1}{6}x - \frac{40}{3} & x \geq 40 \end{cases} \qquad (4\text{-}28)$$

which is shown as Fig. 4-34. Figure 4-35 presents five circuits which will generate this function. The circuit of Fig. 4-35*a* uses a shunt-limiter circuit in the input, and Fig. 4-35*d* uses a shunt limiter in the feedback.

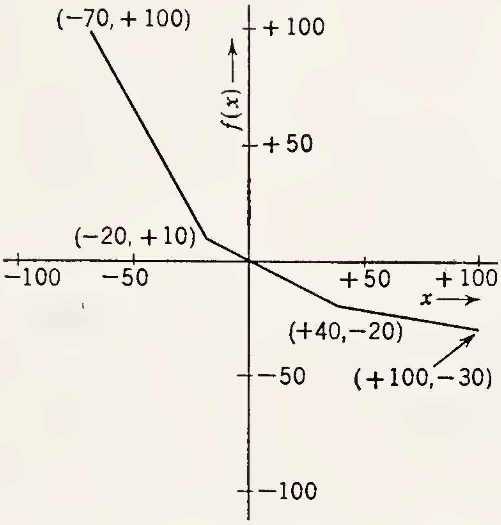


FIG. 4-34. Simple straight-line function.

These circuits are therefore governed by the same relationships that were described in Sec. 4-2. The circuits of Fig. 4-35*b* and *c* are of the series-output and series-input type, respectively. All four of these circuits have the disadvantage of having a diode conducting during the zero region. This problem is eliminated by the circuit of Fig. 4-35*e*, which uses one diode to shunt the input and the other in series. It will be noticed that, as the diodes in a shunt input or a series output network conduct, the gain is decreased. In a series input or shunt output the opposite is true. Thus, by combining shunt-input and series-input techniques, it is possible to generate any three-segment monotonic function and still have the slope for zero input independent of diode characteristics. If two amplifiers are used, it is possible to remove the restriction that the function be monotonic. This technique may be expanded to any number of segments by including more shunt or series diodes. This approach to generating arbitrary functions is considered in Chap. 7.

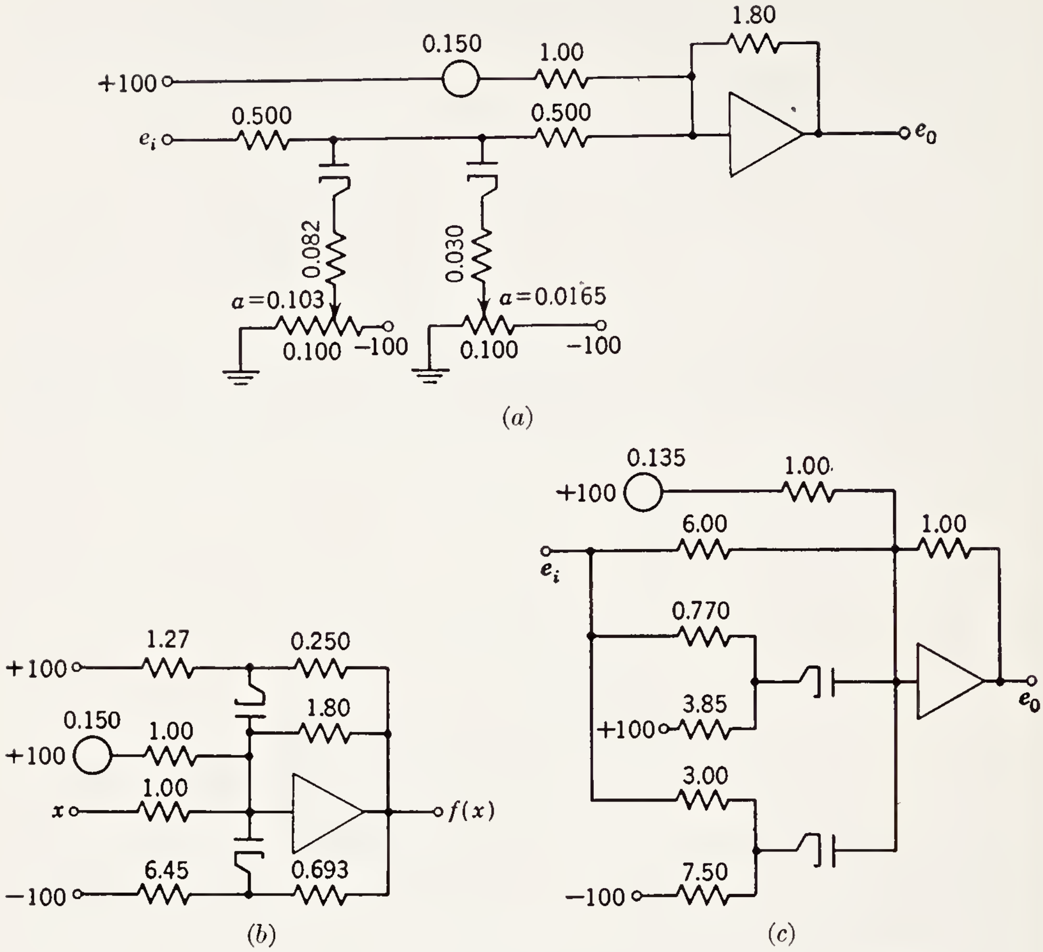


FIG. 4-35. Circuits for approximating a simple straight-line function.

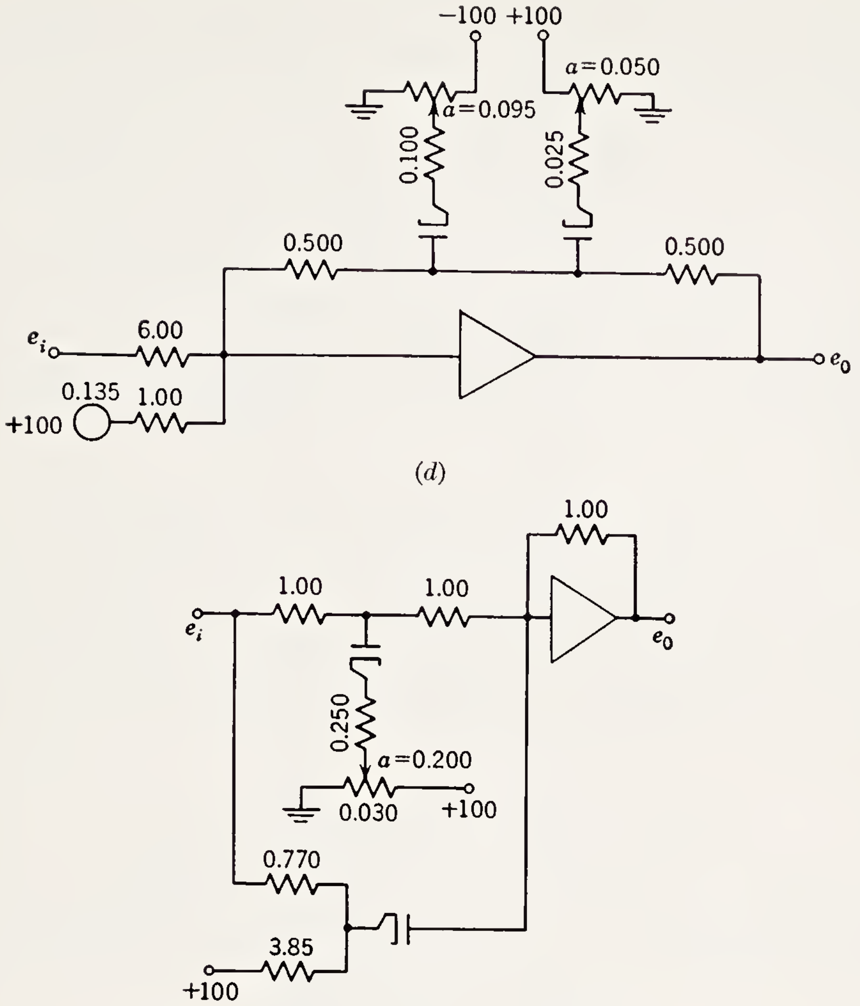
4-9. SIMULATION OF AN ACCELEROMETER

The following example of the simulation of an accelerometer will illustrate how diode circuits can be used in practice. The example also demonstrates how these circuits can be combined with little or no loss in accuracy. Use is made of the techniques of Chap. 3 for best utilization of equipment.

It is desired to simulate an accelerometer having the following physical characteristics:

Maximum range $\pm 2.5g$.
Static friction $0.05g$.
Backlash 0.4 per cent or $0.02g$.
Damping ratio 0.6 to 0.7.
Natural frequency 9 cps.

It is assumed that the physical characteristics of the accelerometer can be represented by a second-order system with a natural frequency of 9 cps



(d)

(e)

Fig. 4-35 (*Continued*)

and a damping ratio between 0.6 and 0.7, with the static friction and backlash superimposed.

Selecting a scale factor for our acceleration of 1 volt = 1 ft/sec², then the maximum value of acceleration A will be 80.5 volts. The backlash can be simulated using the circuit of Fig. 4-20, letting $C_1 = C_2 = 0.10\ \mu$f. We shall keep the resistance R_2 small to reduce the effect of the lag. The width of the backlash loop is $0.02g = 0.644$ ft/sec², so we have from Eq. (4-21) that

$$0.644 = \frac{200R_2}{R_1}$$

Let $R_2 = 0.02$ megohm; then $R_1 = 4/0.644 = 6.22$ megohms. The static friction can be simulated by means of the dead-space circuit of Fig. 4-16*a*. Let $R_2 = 0.02$ megohm; then $R_1 = 1.24$ megohms. The second-order

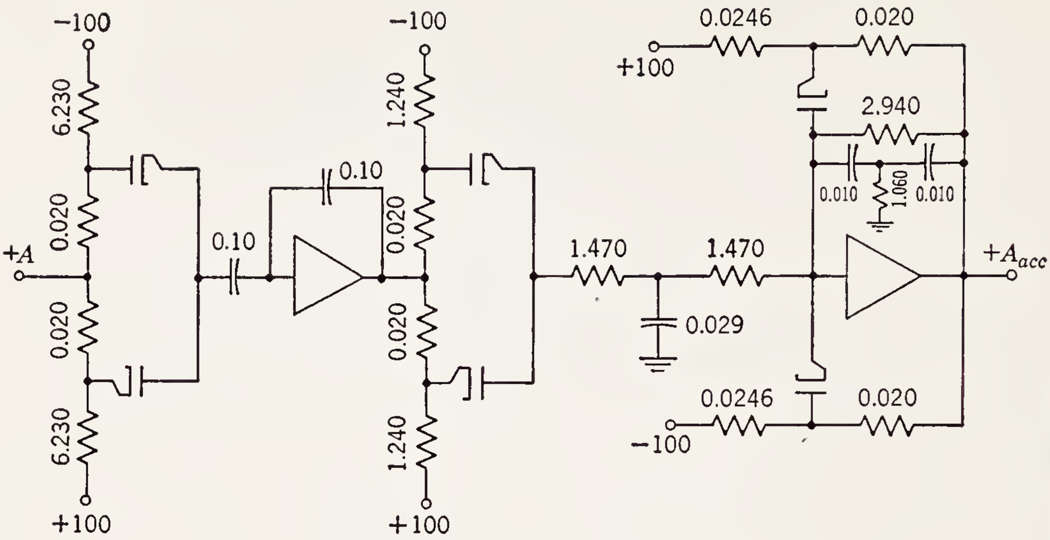


FIG. 4-36. Circuit for simulating an accelerometer.

characteristic can be obtained by using the second-order lag circuit shown in Fig. 3-18. It is desired that the over-all gain of the accelerometer circuit be unity; therefore the resistors R_1 and R_3 should be selected so that $2R_1 = R_3$. We have the relations

$$R_2R_3C_2^2 = \frac{1}{\omega_n^2} = \frac{1}{(9 \times 2\pi)^2} = 3.12 \times 10^{-4}$$

Let $C_2 = 0.01\ \mu\text{f}$; then $R_2R_3 = 3.12$ megohms. The condition of unity gain demands that $2R_1 = R_3$, and the damping ratio demands that

$$2R_2C_2 = \frac{2\xi}{\omega_n} = \frac{(2)(0.6)}{(9)(2\pi)}$$

Thus we have four equations in four unknowns which can be solved for the desired circuit constants. These are

$$R_1C_1 = 0.04R_2$$
$$R_2R_3 = 3.12$$
$$2R_1 = R_3$$
$$0.02R_2 = \frac{1.20}{5.65} = 0.0212$$

Therefore

$$R_2 = 1.06 \text{ megohms}$$
$$R_3 = \frac{3.12}{R_2} = \frac{3.12}{1.06} = 2.94 \text{ megohms}$$
$$R_1 = \frac{R_3}{2} = 1.47 \text{ megohms}$$
$$C_1 = \frac{0.04R_2}{R_1} = \frac{0.04 \times 1.06}{1.47} = 0.029\ \mu\text{f}$$

It is possible to combine the lag circuit and the dead-space circuit onto one amplifier with little loss in accuracy. The range limitations can be provided with a series output limiter which can be combined with the other circuits on the final amplifier. Referring to Fig. 4-12, if we let $R_2 = 0.02$ megohm, we have

$$R_1 = \frac{(0.02)(\pm 100)(2.94)}{(\pm 80.5)(2.96)} = 0.0246 \text{ megohm}$$

The final circuit for simulation of the accelerometer is given in Fig. 4-36. It should be mentioned that, while the techniques of combining circuits demonstrated here decrease the accuracy somewhat, the final circuit displayed gave very satisfactory results in an actual aircraft simulation.

4-10. THE IDEALIZED DIODE CIRCUIT AND ITS APPLICATION

Where extreme accuracy is required in diode circuits, it is possible to idealize the characteristics of a diode by means of the circuit of Fig. 4-37.

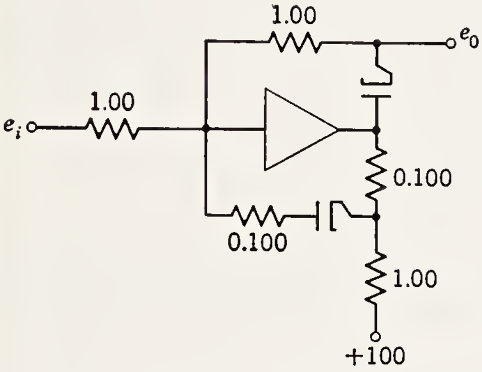


FIG. 4-37. Idealized diode circuit.

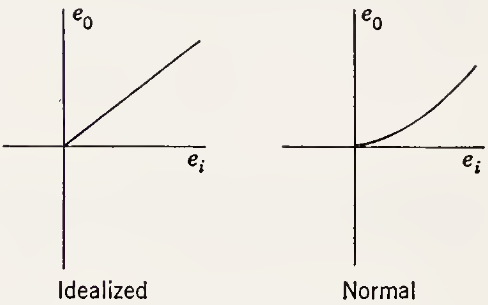


FIG. 4-38. Performance of normal and idealized diode.

The amplifier is essentially used as a comparator to provide a high bias on the diode as soon as the input changes sign. The second diode is simply a one-sided series output limiter to prevent overloading of the amplifier. This circuit gives an output around zero, which, as shown in Fig. 4-38, is in marked contrast to the output in the zero region of the usual diode. If the diodes are reversed and the polarity of the bias changed, the output will be the mirror image of the curve shown. This idealized diode circuit finds many uses where precision around zero diode bias is desired, and it can be used in place of a normal diode in many diode circuits. In general, it requires more amplifiers to do a given job using this technique, but this is not always true. An example is the precision absolute-value circuit shown in Fig. 4-41, which uses no more amplifiers than the usual circuit.

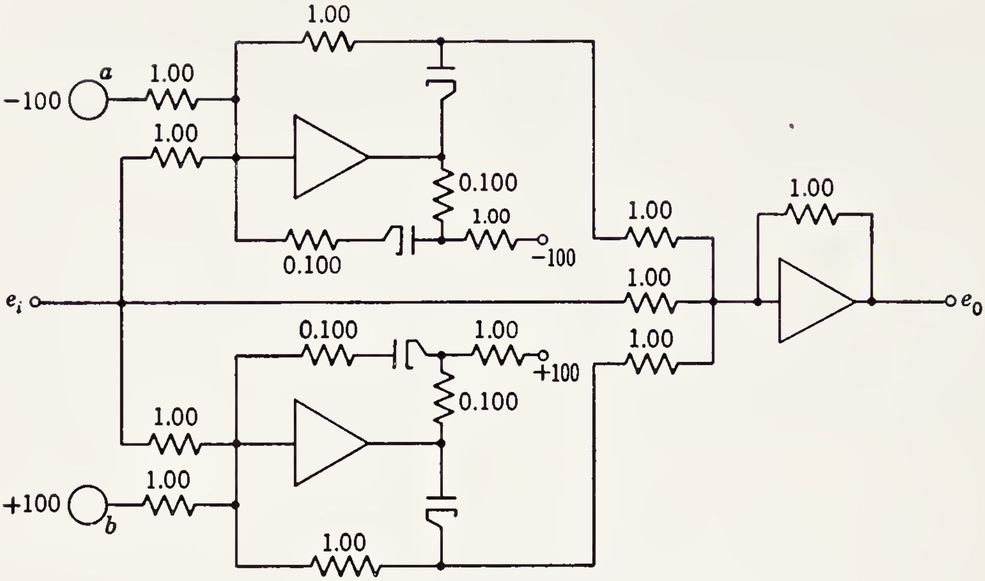


FIG. 4-39. Precision series input limiter.

The idealized-diode technique can be utilized to give excellent limiter performance. Figure 4-39 shows the circuit of such a limiter. The output is given by the following relationships:

$$e_o = \begin{cases} -a & \text{for} \quad e_i \geq a \\ -e_i & \text{for} \ -b \leq e_i \leq a \\ b & \text{for} \quad e_i \leq -b \end{cases}$$

This limiter has excellent linear operation, extremely sharp cutoff characteristics, and limit stops which are flat within the accuracy of virtually

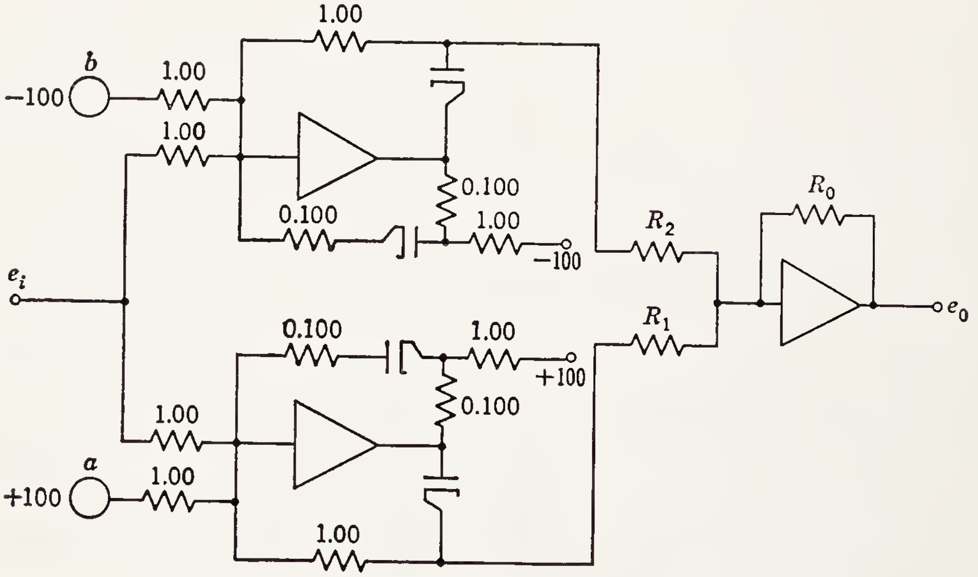


FIG. 4-40. A precision dead-space circuit.

all computer measuring devices. It is extremely easy to adjust and can be used to limit between values of either the same or the opposite sign.

An excellent dead-space circuit can be formed using idealized diode methods. This circuit, illustrated in Fig. 4-40, possesses very sharp cutoff characteristics and is completely general. Its output is governed by the following relationships:

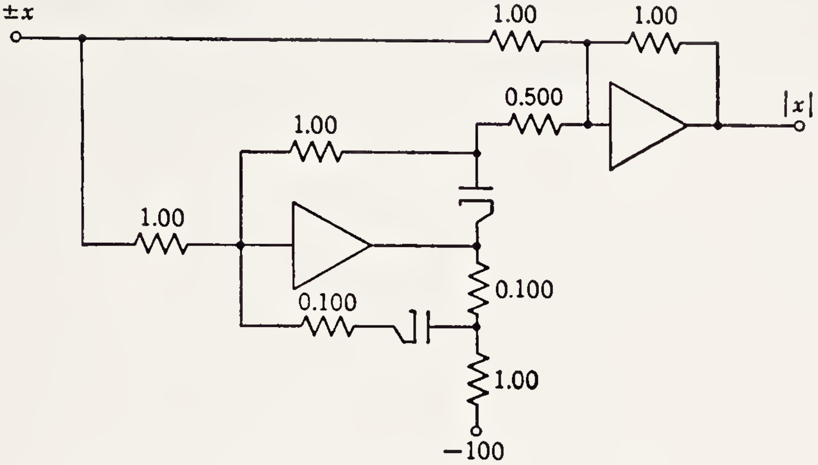


FIG. 4-41. Precision absolute-value circuit.

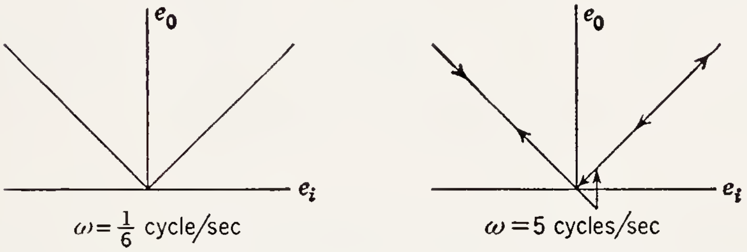


FIG. 4-42. Precision absolute-value circuit output.

$$e_o = \begin{cases} \dfrac{R_0}{R_1}(e_i + a) & e_i \le -a \\ 0 & -a \le e_i \le b \\ \dfrac{R_0}{R_2}(e_i - b) & e_i \ge b \end{cases}$$

One of the most common uses of the idealized diode is to provide a precision absolute-value circuit. Figure 4-41 shows a circuit for this operation which can be easily adapted to any type of patch board in common use. This circuit gives an extremely good calculation of absolute value for low-frequency inputs but does show a slight switching transient for high-frequency applications. Figure 4-42 shows the output of this circuit when driven by sine-wave inputs with the frequencies of ⅙ and 5 cps.

This type of transient is present in all idealized diode circuits and is a function of the diode characteristics, the gain of the amplifier, and the magnitude of the bias applied to the diode of the series output limiter. The duration of the spike is the time required to drive the output of the amplifier from zero to the bias value. The frequency response increases greatly as the bias is lowered, but rounding appears as the bias approaches zero. However, even with the bias voltage as low as 1 volt, the cutoff characteristics are much superior to those of the ordinary diode circuit.

PROBLEMS

4-1. Given the circuit of Fig. P4-1, prove that

$$e_0 = \frac{R_L}{r(R_i + R_L) + R_iR_L}(re_i + R_iaE)$$

where

$$r = R_D + a(1 - a)R$$

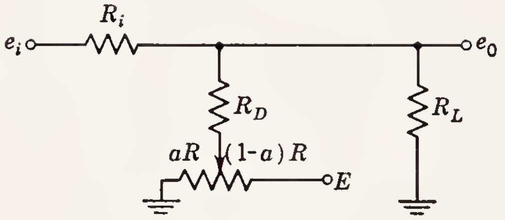


FIG. P4-1.

4-2. A system is governed by the differential equation

$$\ddot{x} + 2\dot{x} + 10x = F(t) \qquad \dot{x}(0) = 0 \qquad x(0) = 0$$

for small oscillations. In addition the system has the following rate and position constraints:

$$|\dot{x}| \leq 20 \qquad |x| < 40$$

Set up a circuit to solve this problem when

(a) $F(t) = A \qquad |A| \leq 50$

(b) $F(t) = at \qquad |a| \leq 20$

(c) $F(t) = |B \sin \omega t| \qquad 0 \leq \omega \leq 20,\ |B| \leq 50$

(d) $F(t) = \dfrac{B(\sin \omega t + |\sin \omega t|)}{2}$

where ω and B have the same limits as in (c). *Hint:* In this case

$$F(t) = \begin{cases} B \sin \omega t & \sin \omega t \geq 0 \\ 0 & \sin \omega t < 0 \end{cases}$$

(e)
$$F(t) = \begin{cases} B \sin \omega t + b & B \sin \omega t \leq -b \\ 0 & -b \leq B \sin \omega t \leq b \\ B \sin \omega t - b & B \sin \omega t \geq b \end{cases}$$

4-3. Set up a circuit to determine the oscillations of a system whose equation of motion is $m\ddot{x} + F(x) = 0$, where $F(x)$ has the form

(a) $$F(x) = \begin{cases} 0 & |x| \leq a \\ 2(x - a) & |x| \geq a \end{cases}$$

(b) $$F(x) = \begin{cases} x + a & x > 0 \\ x - a & x < 0 \end{cases}$$

(c) $$F(x) = \begin{cases} 2x + a & -a > x \\ x & -a \leq x \leq a \\ 2x - a & a < x \end{cases}$$

Use the initial conditions $x = a$ and $\dot{x} = 0$ at $t = 0$.

4-4. A rocket engine burns in such a manner that its thrust is given by the curve

$$T = \begin{cases} 1{,}000(4t - t^2) & 0 \leq t \leq 4 \\ 0 & t \geq 4 \end{cases}$$

Set up a properly scaled circuit to simulate this engine.

4-5. Set up a circuit for the engine of Prob. 4-4 given by

$$T = \begin{cases} 4{,}000t & 0 \leq t \leq 2 \\ 10{,}000 - 4000t & 2 \leq t \leq 4 \\ 0 & t \geq 4 \end{cases}$$

4-6. The saturation characteristics of an amplifier in a servo system can be simulated by the following function:

$$e_o = \begin{cases} \mp 37.5 & |e_i| \geq 60 \\ \pm 22.5 - \frac{1}{4}e_i & 30 \leq e_i \leq 60 \\ -e_i & |e_i| \leq 30 \end{cases}$$

Set up a one-amplifier circuit to simulate this function.

4-7. Given inputs of $+x$ and $-y$, where $|x| \leq 50$ and $|y| \leq 50$, set up a circuit, using only three amplifiers, to find $\int \frac{(|x| + |y|)\,dt}{2}$. Use idealized-diode techniques.

4-8. It is desired to limit the output of a summer to values between -15 and $+35$ volts. Set up circuits to perform this operation, using (*a*) a shunt input limiter; (*b*) a series input limiter; (*c*) a series output limiter.

4-9. Given the input of -200θ, θ in radians, set up a one-amplifier two-diode circuit to approximate 200 sin θ to at least 1 per cent absolute accuracy over the range $-\frac{1}{2} \leq \theta \leq +\frac{1}{2}$.

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2. "Role of Diodes in an Electronic Differential Analyzer, GER-4798," Goodyear Aircraft Corporation, Akron, Ohio, 1952.

3. Korn, G. A., and T. M. Korn: "Electronic Analog Computers," 2d ed., McGraw-Hill Book Company, Inc., New York, 1956.

CHAPTER 5

Function Relays

5-1. INTRODUCTION

In the previous chapter a variety of electronic switching devices were considered in some detail. In this chapter another approach to the problem of switching will be presented: the use of relay circuits. A relay can be defined as an electromechanical device which utilizes the variation of current in one electric circuit to control a switching operation in another. We shall not attempt a discussion of the design requirements for a relay suitable for analog computers, since most present-day machines are equipped with such relays. We shall, instead, present a discussion of the relay comparator, which is the basic unit of relay switching, and then consider a variety of circuits employing this device. We shall conclude with a discussion of the relative merits of diode and relay methods.

5-2. THE RELAY COMPARATOR

Just as the biased diode is the basic element of electronic switching devices, so the relay comparator is the basic element of the electromechanical switches used on an analog computer. The relay comparator performs the switching operation by comparing two input voltages and engaging or de-energizing a relay, depending on whether the sum of the two voltages is positive or negative. The comparator consists of an amplifier with a relay in the plate circuit of the output tube. The amplifier is designed so that, when the input grid is positive or zero, no current flows in the plate circuit of the output tube, and the relay is therefore de-energized. When the input grid becomes negative, the plate circuit of the output tube conducts heavily, energizing the relay. Figure 5-1 shows a simplified diagram of a one-tube relay comparator. The relay shown in Fig. 5-1 has two sets of contacts; more may be used if desired. The contacts are marked NC for "normally closed" and NO for "normally open." Where three or more sets of contacts are available, one set is often used to energize a signal lamp, which indicates the position of the relay arm.

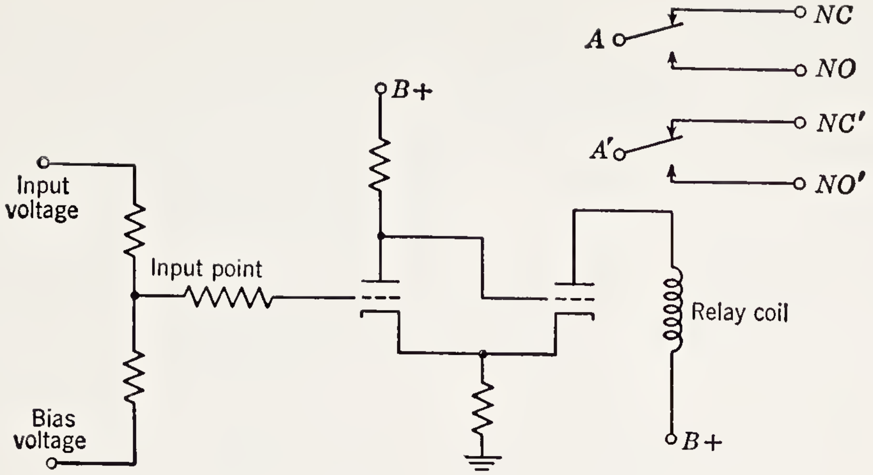

Fig. 5-1. Simple relay comparator circuit.

On many computers the relays are wired onto the patch board directly with no relay amplifier included. When this is done, it is necessary to use an amplifier from the problem board to complete the relay comparator. To achieve this, a diode comparator, such as that of Fig. 4-28, is used to energize the relay. This gives the relay comparator circuit shown in Fig. 5-2.

For simplicity in diagramming, we shall use the symbol of Fig. 5-3 to indicate all types of relay comparators. The relay is closed until the input voltage to the + terminal is positive with respect to that of the − terminal, at which time the relay opens.

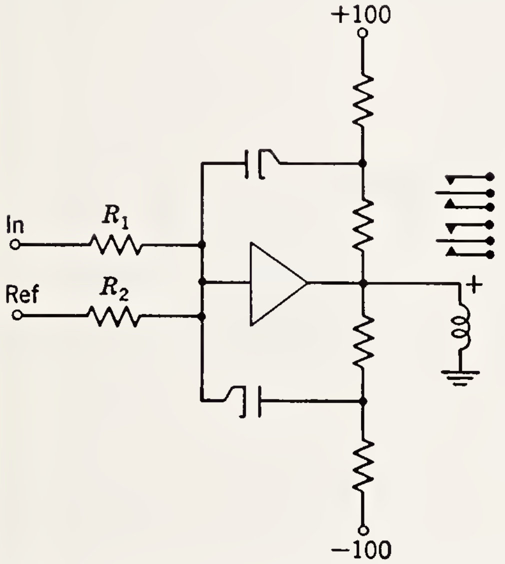

Fig. 5-2. Relay comparator circuit using an operational amplifier.

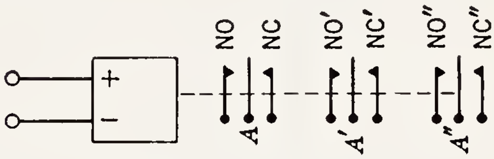

Fig. 5-3. Symbolic representation of the relay comparator.

5-3. THE USE OF RELAYS IN COMPUTER CIRCUITS

In this section we shall show relay circuits which perform similar functions to those of the diode circuits of Chap. 4. This presentation is not complete but will serve as a guide to the interested reader.

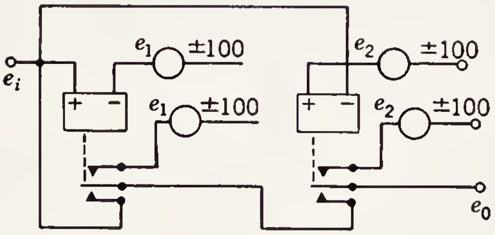

FIG. 5-4. Simple relay limiter circuit.

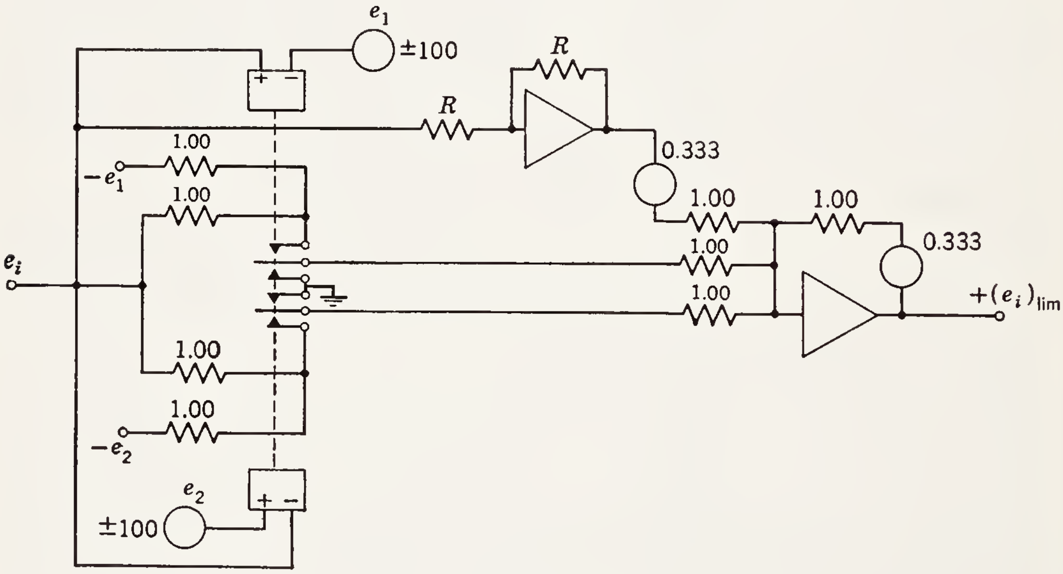

FIG. 5-5. An improved relay limiter circuit.

The relay comparator can be used to limit the output of a summing amplifier in several ways. One of the simplest circuits for this purpose is given in Fig. 5-4. In this circuit the upper limit is given by e_1, the lower limit by e_2. It should be noticed that the loading on the potentiometers connected to the relays changes when the relays open. Thus, those potentiometers must be calibrated when the relays are in the "open" position. In general, it is desirable to plan relay circuits so that the voltages on the relay contacts are near zero at the time of switching. This greatly reduces the transient during the switching operation. In the circuit of Fig. 5-4, this has not been done, and, in the case of the upper limit, both contacts are at $+a$ volts at the time of switching. During the switching there will be a very short period when the arm is not touching either contact and, therefore, the potential on the arm will fall to zero. Hence, a "spiked" transient results. This disadvantage is eliminated in the circuit of Fig. 5-5. This circuit has relationships

$$e_o = \begin{cases} e_1 & e_i \geq e_1 \\ e_i & e_2 \leq e_i \leq e_1 \\ e_2 & e_i \leq e_2 \end{cases} \tag{5-1}$$

The relay comparator can also be used to limit an integrator. The only problem in performing this task is to provide a means of allowing

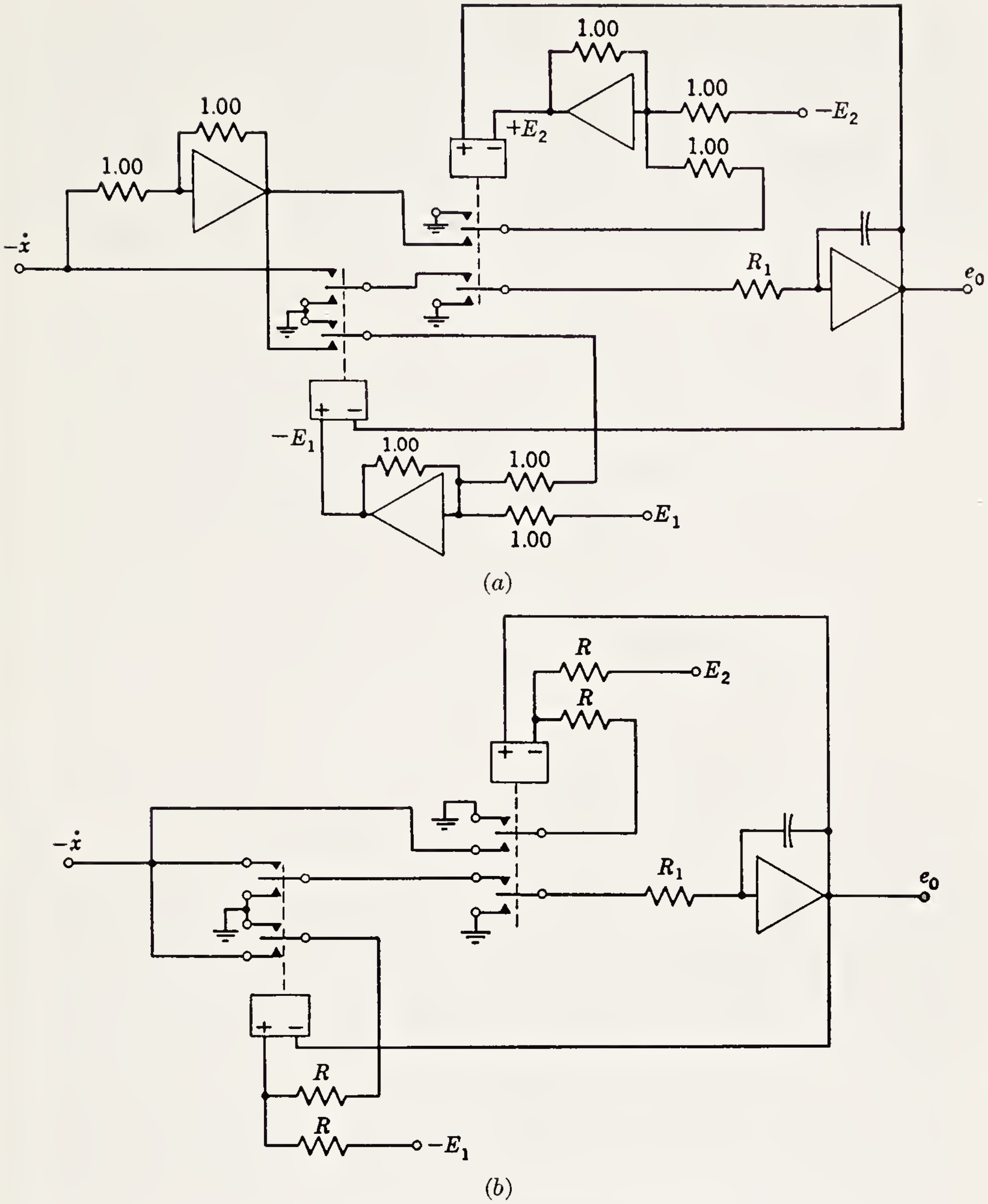

Fig. 5-6. (*a*) Relay limiting of an integrator. (*b*) Relay limiting of an integrator using summation on the comparator input.

the output to back off from the limit value. This problem is not encountered with diodes since the transition from conduction to nonconduction is gradual. Figure 5-6*a* presents a circuit which can be used to limit an integrator. The output of this circuit is given by

$$e_o = \begin{cases} -E_1 & -E_1 \leq x \\ x & -E_1 \leq x \leq E_2 \\ E_2 & x \geq E_2 \end{cases} \tag{5-2}$$

Depending upon the type of relay comparator used, it may be possible to reduce the number of amplifiers required. If summation on the input of the comparator is possible without great loss of gain, the circuit of Fig. 5-6*b* may be used. As an example, the latter circuit may be used with the comparator of Fig. 5-2, while its applicability with a comparator of a design similar to that of Fig. 5-1 would be a function of the resistance used in the comparator. Assuming no loss of gain in the summation, the behavior of both these circuits is identical. In the range between the limit stops the operation of the integrator is unimpaired. When either limit is reached, the relay opens and the output remains constant until the sign of the input changes. This causes a change in the bias of the relay, and the relay closes, thus permitting normal integrator operation again.

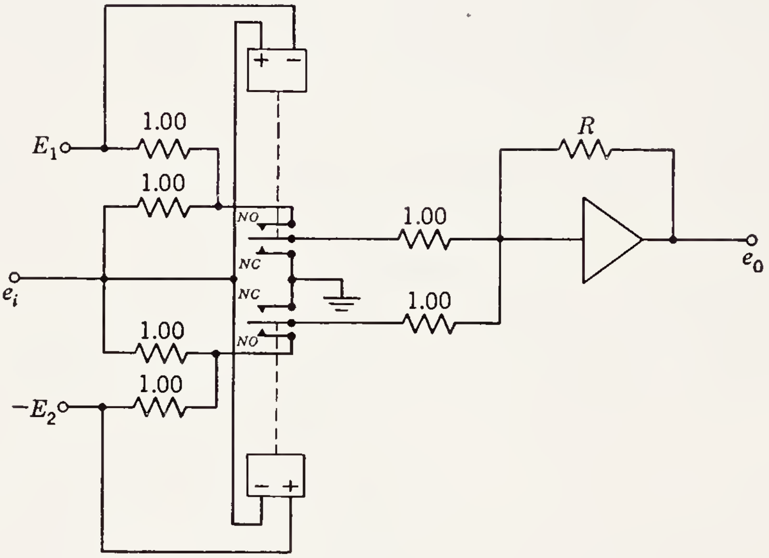

FIG. 5-7. A dead-space circuit using relays.

The phenomenon of the dead space can also be easily simulated using relays. The circuit of Fig. 5-7 shows a possible approach to this problem. This circuit is very similar to the diode circuit of Fig. 4-16*a* in both form and operation.

The circuit of Fig. 5-7 has the relationship

$$e_o = \begin{cases} 0 & -E_2 \leq e_i \leq E_1 \\ \dfrac{-R}{3} e_i & e_i \geq E_1,\ e_i \leq -E_2 \end{cases} \tag{5-3}$$

and is, therefore, limited to the special case when the slopes are constant. This circuit can be easily generalized, however, by making the input resistors variable.

In Sec. 4-5 the diode dead-space circuit of Fig. 4-16*a* was utilized in a closed loop to simulate backlash (see Fig. 4-23). This identical procedure can be used with the circuit of Fig. 5-7. Because of the similarity of the techniques, the circuit will not be detailed here but will be left to the Problems.

The circuit of Fig. 5-8 illustrates a method for simulating absolute value, using relays. This circuit is very simple and is completely analogous to the diode circuit of Fig. 4-25*b*.

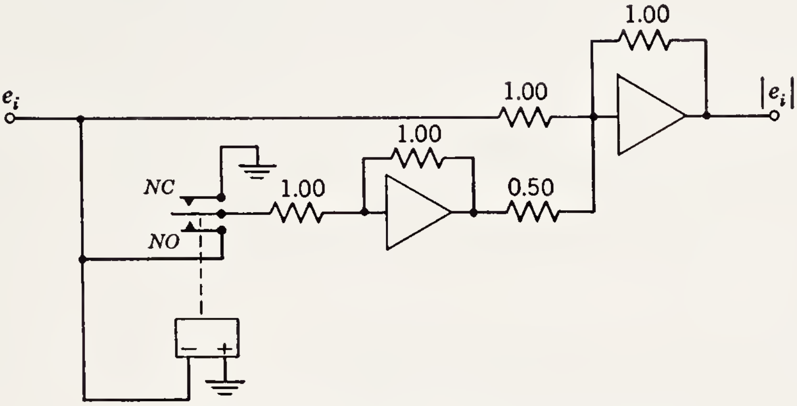

FIG. 5-8. An absolute-value circuit using relays.

5-4. A COMPARISON OF RELAY AND DIODE SWITCHING METHODS

The most fundamental difference between the switching action of diodes and relays is that relays are a discontinuous form of switching, while diodes are continuous. There are only three possible states for the arm of a relay: open, closed, or floating (during switching). Thus, any one path through a relay assumes a resistance value of either zero or infinity. For a diode, however, the transition between conduction and nonconduction is gradual, and a continuous variation of the diode resistance occurs. Hence the relay provides an absolute switching once the switching voltage is reached, where a diode provides an absolute switching only for voltages considerably larger than the switching voltage. The switching characteristics of a relay comparator are expressed in terms of sensitivity and switching time, where the sensitivity is a measure of the input voltage differential required to operate the relay. It is desirable to have the switching time as short as possible, and one would expect that a very sensitive comparator (i.e., a very low differential voltage) would also be desired. However, the noise level prevalent on the computer sets an upper limit to the sensitivity. If the comparator is too sensitive, the relay will make and break contact as a function of the noise and will therefore "chatter." The performance specifications of the relays and relay comparators in use in analog facilities vary widely.

The switching time is generally in the range of 1 to 50 msec, while comparator sensitivity varies from 0.05 to 0.50 volt.

From the previous remarks, we can draw some general conclusions as to the relative merits of relay and diode switching for various types of problems. The greatest advantage of relay methods is that they provide an absolute switching; the greatest disadvantage is that they have a finite switching time. Therefore relays are best suited to those problems where a complete switching of a slowly moving variable is required. Diode methods are preferable where rapid switching is required, where the noise level is high, and where some rounding of the switching is permissible. The stability of diode circuits is generally greater than that of comparable relay circuits, since relays tend to chatter. The idealized diode techniques presented in Chap. 4 have many of the advantages of relays with considerably better frequency response.

PROBLEMS

5-1. Work Prob. 4-2, using relay comparators.

5-2. Work Prob. 4-3, using relay comparators.

5-3. Work Prob. 4-4, using relay comparators.

5-4. Work Prob. 4-5, using relay comparators.

5-5. Set up a relay comparator circuit to simulate backlash, and state the relationships governing the circuit.

5-6. Set up relay circuits for a bistable and a free-running multivibrator.

5-7. Set up a relay circuit to generate the function

$$f(x) = \begin{cases} 0 & x \leq -50 \\ 25 + \tfrac{1}{2}x & -50 \leq x \leq 0 \\ 25 & 0 \leq x \leq 10 \\ 23 + \dfrac{x}{5} & 10 \leq x \end{cases}$$

5-8. Set up a relay circuit to generate the function

$$f(t) = \begin{cases} 0 & 0 \leq t \leq 10 \\ t & 10 \leq t \leq 30 \\ \dfrac{t^2}{30} & 30 \leq t \leq 50 \\ 83.33 & 50 \leq t \leq 100 \end{cases}$$

where t is machine time in seconds.

REFERENCE

Johnson, C. L.: "Analog Computer Techniques," McGraw-Hill Book Company, Inc., New York, 1956.

CHAPTER 6

Implicit-function Techniques

6-1. INTRODUCTION

In Chap. 1 it was pointed out that by using high-gain amplifiers it is possible to generate functions as $Z = x_1/x_2$, $Z = \arctan (x_1/x_2)$, and $Z = \sqrt{x_1^2 + x_2^2}$, etc. In this application the feedback loop of the amplifier is opened, and a device which generates the inverse of the desired relation is introduced into the feedback loop. Thus the feedback loop is closed through a computing element which generates the inverse of the relation desired. In many cases these inverse relations are simple operations such as multiplication, sine generation, etc., whose generation is carried out by techniques already discussed.

Since the computation in this case takes place by substituting a different element in the feedback, there is a possibility that the resulting circuit will be unstable. The necessary but not sufficient condition for stability is: If one traces from the output of the amplifier whose feedback is open, back to its input grid, he must pass through an even number of amplifiers. In the case where instability results it will usually be possible to rearrange the circuit or rescale the function so that the desired functional relationship is obtained by a stable circuit.

In this chapter the fundamental concept used in all implicit circuits is developed. Application of this concept is then made to obtain circuits which perform the operations of division, square root, arcsine, and arctangent. Examples illustrating the programming of equations requiring implicit techniques are also given. The implicit-function technique offers a general method for performing these operations. In most cases alternative methods exist for realizing the mathematical operations. Some of the more useful of these methods are presented for purposes of comparison.

6-2. THE USE OF HIGH-GAIN AMPLIFIERS FOR IMPLICIT COMPUTATION

The use of high-gain d-c amplifiers for implicit computation may be considered as a generalization of the operational amplifier technique and is not restricted to linear operations. Figure 6-1 shows a computer setup which is useful for solving equations of the type

$$ke_1 + F(e_2,e_3, \ . \ . \ . \ ,e_n,e_o) = 0 \qquad (6\text{-}1)$$

for the variable e_o, where e_1, e_2, . . . , e_n are other computer voltages. The function $F(e_2,e_3, \ . \ . \ . \ ,e_n,e_o)$ is restricted to those operations which do not involve a sign inversion. For those functions which have a sign inversion an additional amplifier is required to preserve the stability of the circuit. Writing the nodal equation for the input grid of the amplifier, we obtain

$$\left(\frac{e_o}{\mu} - e_1\right)\frac{1}{R_1} + \left[\frac{e_o}{\mu} - F(e_2,e_3, \ . \ . \ . \ ,e_n,e_o)\right]\frac{1}{R_2} = 0$$

If in this equation we let $\mu \rightarrow \infty$, there is obtained

$$\frac{e_1}{R_1} = -\frac{F(e_2,e_3, \ . \ . \ . \ ,e_n,e_o)}{R_2} \qquad (6\text{-}2)$$

which is the required solution of the implicit equation. Equation (6-2) is the fundamental equation of all implicit-function circuits.

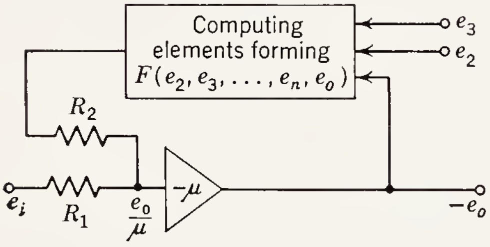

Fig. 6-1. Implicit-function circuit.

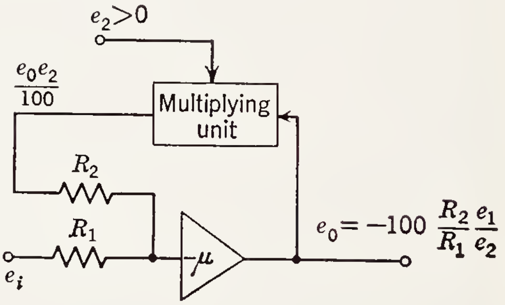

Fig. 6-2. The division circuit.

6-3. THE DIVISION CIRCUIT

Consider the division of two voltages e_1 and e_2. Following the general procedure outlined in Secs. 6-1 and 6-2, this can be accomplished by means of connecting a multiplier as the feedback element of a high-gain amplifier as illustrated by Fig. 6-2. The multiplying unit in the feedback may consist of any analog multiplier such as a servomultiplier or a pulse-height pulse-width multiplier. In all these devices

$$F(e_2,e_3, \ . \ . \ . \ ,e_n,e_o) = \frac{e_o e_2}{100}$$

assuming 100 volts operating range; so

$$\frac{e_1}{R_1} = \frac{-e_o e_2}{100R_2}$$

or

$$e_o = -100 \frac{R_2}{R_1} \frac{e_1}{e_2} \qquad (6\text{-}3)$$

which is the desired quotient. It will be noted that the ratio R_2/R_1 determines the scaling of the quotient e_1/e_2. In the particular case of the servomultiplier one possible circuit is as shown in Fig. 6-3. It should be noted that, in this circuit, the voltage representing the numerator y can take on both positive and negative values. The voltage x representing the denominator can take on only positive values. However, since y can take on both positive and negative values, the negative value of x or $-x$ is required to provide the feedback when y is positive. In general this may require the use of an additional inverting amplifier. Also this circuit will exceed the 100-volt operating limit as $x \to 0$. Thus, if $y = 100$ volts, the minimum value that x can have and still be within the operating limits of the computer is $100R_2/R_1$ volts. Division, therefore, requires special consideration in selecting scale factors. In some computers difficulty may be experienced with the stability of the circuit of Fig. 6-3. This difficulty arises because the servomotor with its poor response characteristics is an element in the feedback loop of the amplifier. An alternative circuit may be employed to perform division, using a servomultiplier. This circuit is illustrated by Fig. 6-4. It differs from

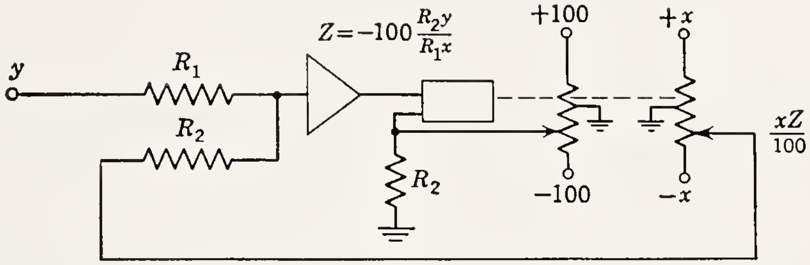

FIG. 6-3. Servomultiplier division circuit.

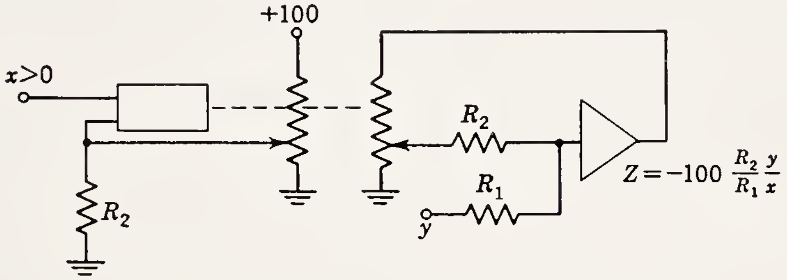

FIG. 6-4. Alternative servomultiplier division circuit.

the circuit of Fig. 6-3 in that the roles of the numerator and denominator terms in the feedback multiplication circuit are interchanged. In so doing the servo response characteristics are removed from the feedback loop of the amplifier. As a result the circuit of Fig. 6-4 tends to be more stable than the circuit of Fig. 6-3. This second circuit has the additional advantage that only $+x$ is required.

An additional nonimplicit division circuit using a servomultiplier is commonly used (see Fig. P1-8). This circuit, shown as Fig. 6-5, does not make use of the implicit-function technique but in the interest of easy reference is presented here. In this circuit plus and minus the denominator voltage is placed on opposite ends of the follow-up potentiometer.

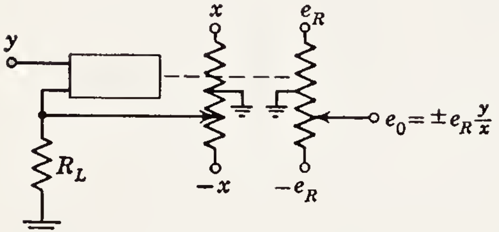

FIG. 6-5. Servo division circuit using follow-up potentiometer.

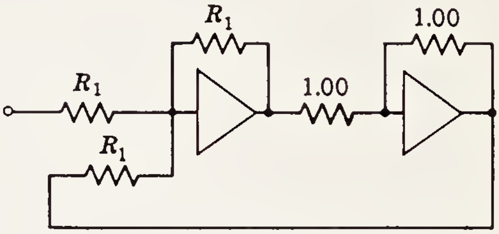

FIG. 6-6. Unstable feedback around two amplifiers.

If $\pm e_R$ is placed across one of the multiplying potentiometers, then a voltage

$$e_o = \pm e_R \frac{y}{x} \tag{6-4}$$

will be obtained from the arm of that potentiometer. A disadvantage of this circuit is that the servo loop gain decreases as the follow-up voltage decreases. This leads to poor servo response for small values of x. The circuit will generally give satisfactory results for $x > 30$ volts.

A circuit for performing division exists which is similar in operation to the implicit circuits but does not make use of a high-gain amplifier. Consider the connection of two summing amplifiers in series, with the output of the second amplifier fed back as an input to the first amplifier, as in Fig. 6-6. Ordinarily such an arrangement of amplifiers would be unstable, since the second amplifier acts as a positive feedback path around the first amplifier. If we substitute this combination in the place of a high-gain amplifier, it can be seen that this circuit will perform the same function in implicit-function generation as was previously performed by the high-gain amplifier. To illustrate the use of the circuit, consider its application in the servomultiplier division circuit of Fig. 6-7. To analyze the performance of the circuit, let us consider the output of

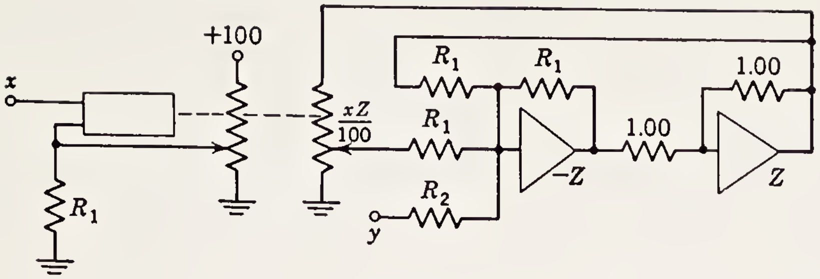

FIG. 6-7. Two-amplifier division circuit.

amplifier 1. The voltage out of this amplifier is

$$-Z = -\left(Z + \frac{R_1}{R_2}y + \frac{xZ}{100}\right) \tag{6-5}$$

Solving Eq. (6-5) for Z, we obtain for the voltage out of amplifier 2

$$Z = -100\frac{R_1}{R_2}\frac{y}{x} \tag{6-6}$$

The unstable combination of amplifiers 1 and 2 has thus functioned in place of the high-gain amplifier previously used. This circuit is of value in those computers where it is difficult to disconnect the amplifier feedback and thereby make a high-gain amplifier. Since most of the newer computing equipment has provisions for readily converting the amplifiers to high gain, the circuit is rarely used. In addition the circuit suffers from the fact that if the gain of the feedback from amplifier 2 to amplifier 1 is not unity the over-all gain of the two-amplifier combination is greatly reduced. To illustrate this, let k be the gain for the voltage fed back from amplifier 2 into amplifier 1. That is, $k = R_1/R_1'$, where R_1' is the input resistor supposed not equal to R_1. In this case Eq. (6-5) becomes

$$\begin{aligned} -Z &= -\left(kZ + \frac{R_1}{R_2}y + \frac{xZ}{100}\right) \\ \text{or} \qquad -Z(1-k) &= -\left(\frac{R_1}{R_2}y + \frac{xZ}{100}\right) \end{aligned} \tag{6-7}$$

Now, looking at the two-amplifier combination, the effective gain of the combination is

$$\frac{Z}{(R_1/R_2)y + xZ/100} = \frac{1}{1-k} = \frac{e_o}{e_i} \tag{6-8}$$

For $k = 1$ the gain is infinite, and the required division circuit is realized. However, if in Eq. (6-8) k differs from unity by, say, 0.1 per cent, the gain of the two amplifier combinations is reduced to 1,000. This results in relatively poor accuracy for this circuit.

By using an electronic multiplier a very simple division circuit (Fig. 6-8) can be realized. This circuit may have a tendency to be unstable. If this is so, a small capacitor, say 0.001 to 0.02 μf, can be wired from the output back to the amplifier grid. This will usually result in a stable circuit of relatively high bandwidth.

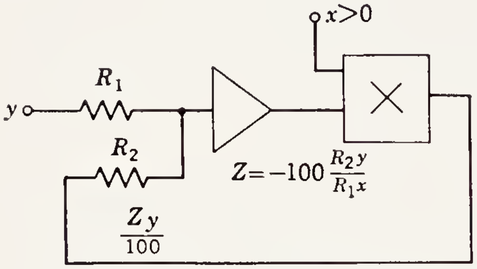

FIG. 6-8. Electronic-multiplier division circuit.

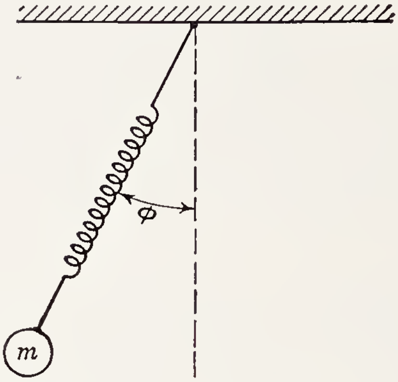

FIG. 6-9. The elastic pendulum.

Example 1. As a simple problem involving division, let us consider the equation of motion of the elastic pendulum of Fig. 6-9. Let m be the mass of the bob, l_0 the length of the pendulum in the absence of load, k the spring constant, and g the acceleration of gravity. The equations of motion for the elastic pendulum can be established most simply through application of energy principles and Lagrange's equations of motion. By this method the equations are found to be

$$\begin{aligned} \ddot{\phi} &= -\frac{g}{r}\sin\phi - \frac{2\dot{\phi}\dot{r}}{r} \\ \ddot{r} &= r\dot{\phi}^2 - \frac{k}{m}(r - l_0) + g\cos\phi \end{aligned} \tag{6-9}$$

Let us consider the specific case where

$l_0 = 2$ ft
$k = 100$ lb/ft
$m = 1$ slug

Substituting into Eq. (6-9), there are obtained

$$\begin{aligned} \ddot{\phi} &= -\frac{32.2}{r}\sin\phi - \frac{2\dot{\phi}\dot{r}}{r} \\ \ddot{r} &= r\dot{\phi}^2 - 100(r - 2) + 32.2\cos\phi \end{aligned} \tag{6-10}$$

To program Eq. (6-10) for the analog computer, we choose the scale factors as follows:

$$\phi < \frac{\pi}{2} \text{ radians}$$

So, computing ϕ in degrees, we let

$$\tfrac{1}{2}\text{ volt} = 1\text{ deg}$$

Assuming that $r < 4$ ft, we compute $25r$, where r is in feet. Nothing is known about the rates of motion, but the following simple calculation will provide a rough estimate: From Eqs. (6-10) we can, on neglecting the products of the rates and taking $\sin\phi = \phi$ and $r = l_0$, obtain the equation

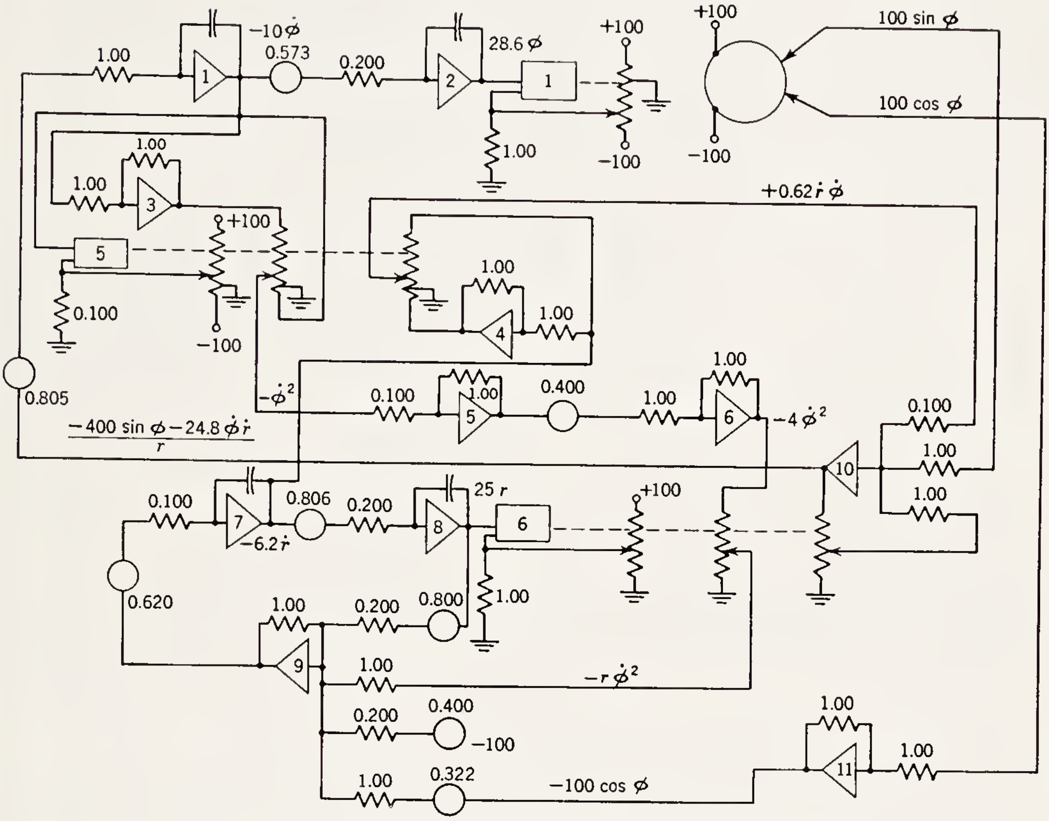

FIG. 6-10. Circuit for solution of the elastic pendulum.

$$\ddot{\phi} = -16.1\phi$$

which has the solution $\phi = A \sin 4.01t$. Differentiating this equation, we obtain $\dot{\phi} = 4.01A \cos 4.01t$, which has a maximum value of 4.01 times the maximum of ϕ. This value is 6.30 radians/sec. To allow a margin of safety, we shall compute $10\dot{\phi}$. Using the same reasoning, a maximum scale factor of 2.50 is obtained for $\dot{r}$. This scale factor turns out to be extremely conservative; therefore, after a process of trial and error, a scale factor of $6.2\dot{r}$ is adopted.

Figure 6-10 shows the completed circuit. The division is accomplished by means of amplifier 10 and servo 6. The circuit of Fig. 6-4 is used for the division. The summing of the two quantities to be divided by r

on amplifier 10 should be noted. It is thus apparent that Eq. (6-3) can be generalized to include the division of n voltages by a single voltage, and hence we obtain

$$e_o = -\frac{100R_2}{e_2}\sum_{i=1}^{n}\frac{e_i}{R_i} \tag{6-11}$$

Example 2. As a second example of a problem which requires division, let us consider the problem of a coin on a smooth table. Suppose that a coin is released from rest with one point of the rim touching a smooth, horizontal table. We shall determine the motion of the center of gravity of the coin.

The forces acting are gravity, Mg down, and the reaction R of the table upward. Thus the center of gravity of the coin descends in a straight line. Let the distance of the center of gravity above the table be denoted by y. Then the dynamical equations are

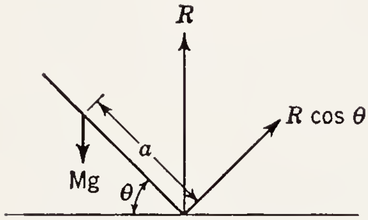

FIG. 6-11. Geometrical relations for falling coin.

$$\begin{aligned} Mk^2\left(\frac{d^2\theta}{dt^2}\right) &= -aR\cos\theta \\ M\frac{d^2y}{dt^2} &= R - Mg \end{aligned} \tag{6-12}$$

where k = radius of gyration. From Fig. 6-11 we have the geometrical equation

$$y = a\sin\theta \tag{6-13}$$

On eliminating R and y, we obtain

$$(k^2 + a^2\cos^2\theta)\frac{d^2\theta}{dt^2} - a^2\sin\theta\cos\theta\left(\frac{d\theta}{dt}\right)^2 = -ag\cos\theta \tag{6-14}$$

Solving for $d^2\theta/dt^2$, there is obtained

$$\frac{d^2\theta}{dt^2} = \frac{a^2\sin\theta(d\theta/dt)^2 - ag}{k^2 + a^2\cos^2\theta}\cos\theta \tag{6-15}$$

The solution is to be obtained under the initial conditions

$$\theta = \alpha \qquad \dot{\theta} = 0 \qquad \text{at } t = 0$$

It is easy to see that $0 < \theta \leq 90$ deg is the range of definition of the variable θ. For the specific case of a 25-cent piece we have the following physical data:

$a = 0.5$ in.
$g = 386$ in./sec^2
$k = \frac{\sqrt{5}\,a}{2} = \frac{\sqrt{5}}{4} = 0.558$ in.

Substituting in the numerical values, we obtain the equation

$$\frac{d^2\theta}{dt^2} = \frac{0.25 \sin \theta (d\theta/dt)^2 - 193}{0.311 + 0.25 \cos^2 \theta} \cos \theta \tag{6-16}$$

To program this equation, we shall make a time transformation to slow the solution down by a factor of 100. To do this, multiply the second derivative of θ with respect to t by 10^4 and the first derivative by 10^2, to obtain

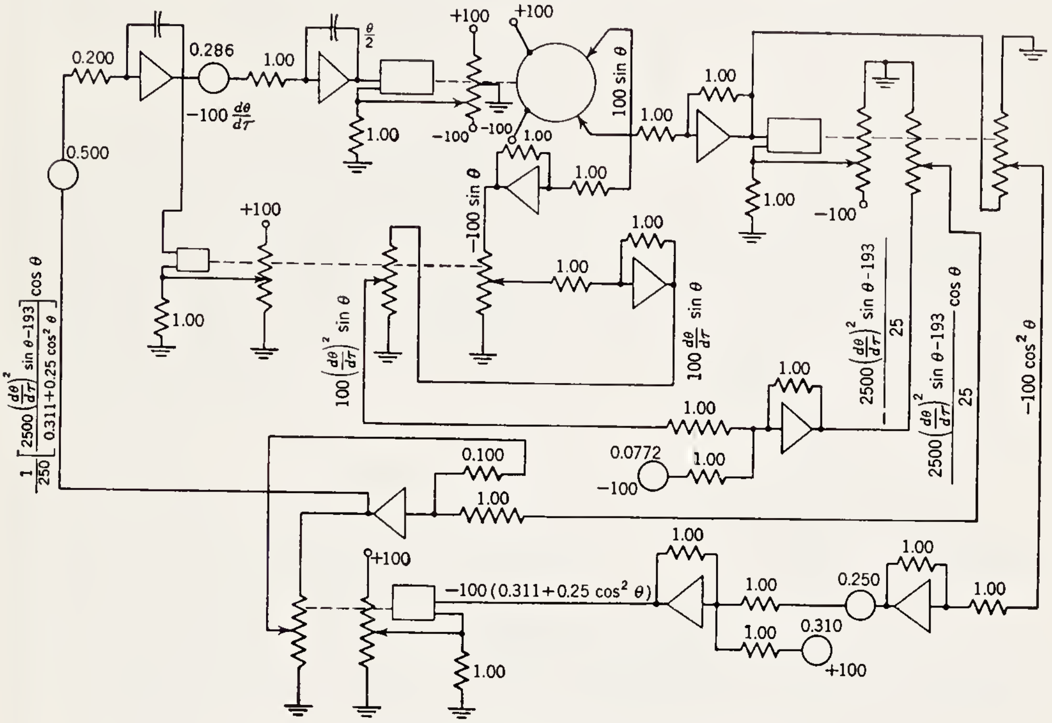
0.200
0.286
1.00
$-100\frac{d\theta}{d\tau}$
$\frac{\theta}{2}$
+100
+100
100 sin θ
1.00
1.00
1.00
−100
−100
1.00
0.500
+100
1.00
−100 sin θ
1.00
1.00
1.00
1.00
−100
$100\left(\frac{d\theta}{d\tau}\right)^2 \sin\theta$
$100\frac{d\theta}{d\tau}\sin\theta$
$2500\left(\frac{d\theta}{d\tau}\right)^2\sin\theta-193$
25
$2500\left(\frac{d\theta}{d\tau}\right)^2\sin\theta-193$
25
cos θ
$-100\cos^2\theta$
$\frac{1}{250}\left[\frac{2500\left(\frac{d\theta}{d\tau}\right)^2\sin\theta-193}{0.311+0.25\cos^2\theta}\right]\cos\theta$
1.00
1.00
0.100
1.00
0.0772
1.00
−100
1.00
1.00
+100
$-100(0.311+0.25\cos^2\theta)$
1.00
0.250
1.00
1.00
0.310
+100
1.00

FIG. 6-12. Circuit for solution of Eq. (6-17).

$$10^4 \frac{d^2\theta}{dt^2} = \frac{2{,}500 \sin \theta (d\theta/dt)^2 - 193}{0.311 + 0.25 \cos^2 \theta} \cos \theta \tag{6-17}$$

Figure 6-12 shows the completed circuit diagram for the simulation of Eq. (6-17).

6-4. THE SQUARE-ROOT CIRCUIT

A voltage proportional to the square root of another voltage may be easily computed by connecting a device which computes the product of a voltage times the absolute value of the voltage as feedback to a high-gain amplifier. Figure 6-13 shows the required circuit. Here $F(e_2, e_3, \ldots, e_o) = e_o|e_o|/100$; hence, on substituting into Eq. (6-2),

$$\frac{e_i}{R_1} = -\frac{e_o|e_o|}{100R_2}$$

For $e_i > 0$ we shall have $e_o < 0$, so that

$$\frac{100R_2}{R_1} e_i = e_o^2$$

and since $e_o < 0$,

$$e_o = -10\sqrt{\frac{R_2}{R_1}}\sqrt{e_i} \tag{6-18a}$$

For $e_i < 0$, we have $e_o > 0$ and

$$e_o^2 = -\frac{100R_2}{R_i} e_i$$

and

$$e_o = 10\sqrt{\frac{R_2}{R_1}}\sqrt{|e_i|} \tag{6-18b}$$

In using this circuit any analog multiplier can be used to perform the multiplication required in the feedback. Circuits for the principal types of analog multipliers are given below.

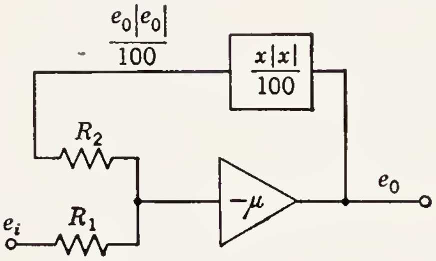

FIG. 6-13. The square-root circuit.

A square-root circuit using a servomultiplier is shown in Fig. 6-14. In this circuit the product $e_o|e_o|/100$ is obtained in the conventional manner. The circuit as shown is for $e_i > 0$; for $e_i < 0$ the square root can be obtained by interchanging the top and bottom of the servo potentiometers and by replacing -100 by $+100$ volts.

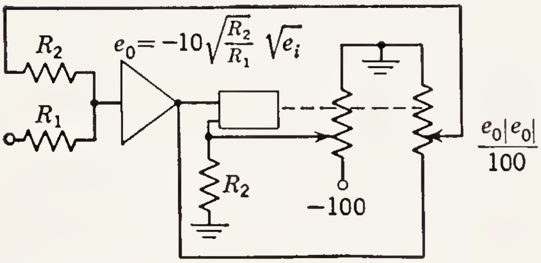

FIG. 6-14. Servomultiplier square-root circuit.

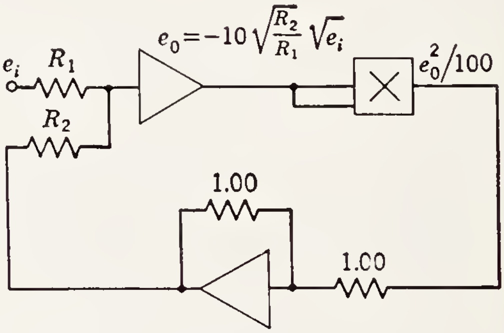

FIG. 6-15. Electronic-multiplier square-root circuit.

In the case of most electronic multipliers an inverting amplifier is required for $e_i > 0$. A square-root circuit using a pulse-height pulse-width electronic multiplier is shown as Fig. 6-15. For the input voltage $e_i < 0$ the inverting amplifier in the feedback of the high-gain amplifier is not required, and the output is $10\sqrt{R_2/R_1}\sqrt{|e_i|}$.

Example. A simple problem which requires the generation of a voltage proportional to the square root of another voltage is the differential equation for the suspension cable. Equation (6-19) is the differential equation for such a cable,

$$\frac{d^2y}{dt^2} = \frac{W}{H}\sqrt{1 + \left(\frac{dy}{dt}\right)^2} \tag{6-19}$$

where W = weight of cable, lb/ft
H = horizontal tension, lb
t = span of cable, ft

and y is related to the sag of the cable. We may program this equation for the computer in terms of the ratio W/H. By varying this ratio we shall generate a family of suspension cables. To plot this curve, we use an integrator whose input is constant to produce a voltage linear with t. If this voltage is properly calibrated, then plotting y versus the output of this integrator produces the desired curve. Figure 6-16 shows the completed circuit.

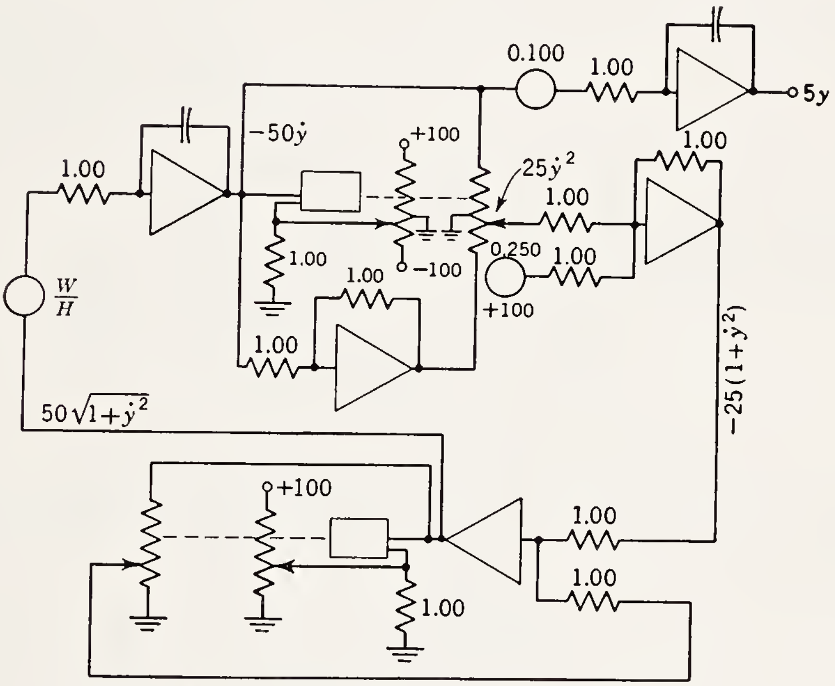

Fig. 6-16. Circuit for solution of the suspension cable.

The square-root circuit can easily be extended to calculate cube, fourth, and fifth roots. The process is illustrated for the cube and fourth roots by Figs. 6-17 and 6-18, respectively. To analyze the performance of the cube-root circuit, we use a method that is generally applicable in the case of implicit circuits. This is: Let the output of the high-gain amplifier be Z; then trace through the circuit, making the required calculations on Z, until one arrives back at the input grid of the high-gain amplifier. Write a nodal equation for all input voltages to this grid as if this point were ground, giving due consideration to having negative feedback round the high-gain amplifier. This procedure is equivalent to using the general formula of Eq. (6-2). For the case of the cube-root circuit the nodal equation is

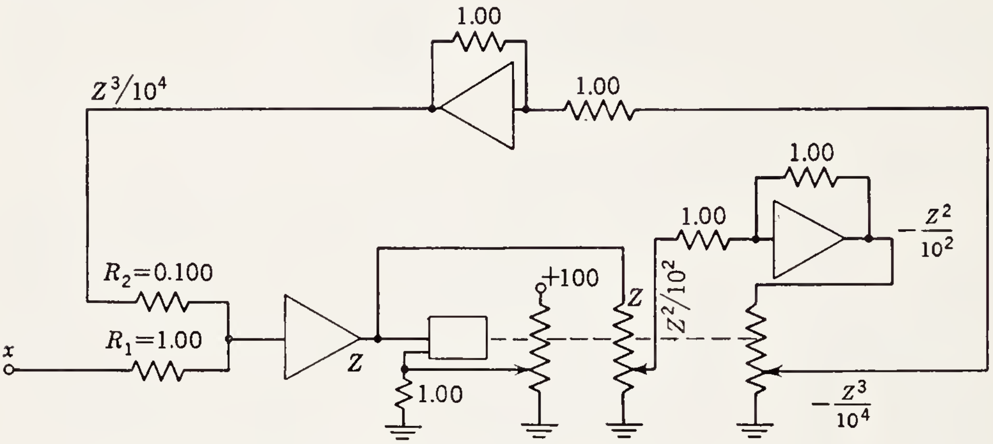

Fig. 6-17. Circuit for calculating the cube root.

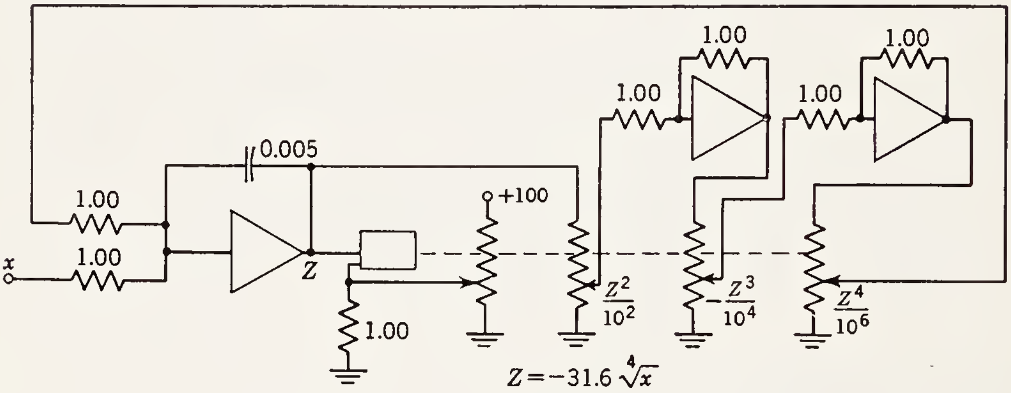

Fig. 6-18. Circuit for calculating the fourth root.

$$\frac{Z^3}{10^4R_2} + \frac{x}{R_1} = 0$$

or

$$Z^3 = -\left(10^4\frac{R_2}{R_1}x\right)$$

and if $R_2 = 0.100$ megohm, $R_1 = 1.00$ megohm, then

$$Z = -10\sqrt[3]{x} \tag{6-20}$$

The calculation of the fourth root may be analyzed similarly. In the case of the fourth root a small capacitor, 0.005 μf, is introduced round the high-gain amplifier to improve stability. The general use of this technique is discussed in Sec. 6-7.

6-5. THE INVERSE-SINE FUNCTION

A voltage which is proportional to the arcsine of another voltage may be easily generated using implicit-function methods. This may be accomplished by connecting a sine-generating device as feedback in a high-gain amplifier, as in Fig. 6-19. Here $F(e_2,e_3, \ldots ,e_n,e_o) = 100 \sin e_o$, and hence

$$\frac{e_1}{R_1} = -\frac{100 \sin e_o}{R_2}$$

or

$$e_o = -\arcsin \frac{e_1 R_2}{100 R_1}$$

and, with $R_1 = R_2$, this gives

$$e_o = -\arcsin \frac{e_1}{100} \tag{6-21}$$

The servo resolver or a diode sine-function generator, described in Chap. 7, may be used to realize the circuit of Fig. 6-19.

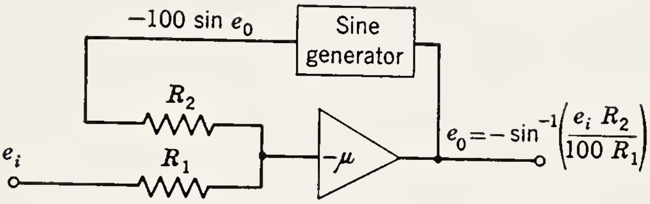

FIG. 6-19. The arcsine circuit.

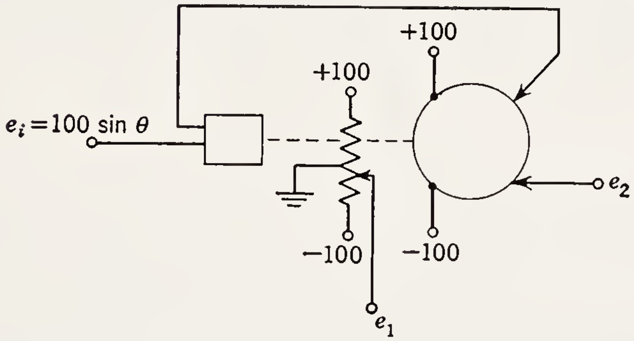

FIG. 6-20. Nonimplicit arcsine circuit.

A nonimplicit-function circuit (Fig. 6-20) using a servo resolver can be employed for generating a voltage proportional to the arcsine of another voltage. In this circuit the feedback for the resolver is closed through the sine potentiometer. If we let the shaft rotate through an angle, say,

ϕ, then the resolver will rotate until a voltage 100 sin ϕ nulls out the error voltage to the servo. Or

$$\begin{aligned} 100 \sin \theta &= 100 \sin \phi \\ \theta &= \phi \end{aligned} \tag{6-22}$$

Thus the arm of a linear potentiometer will read a voltage proportional to the arcsine of the input voltage.

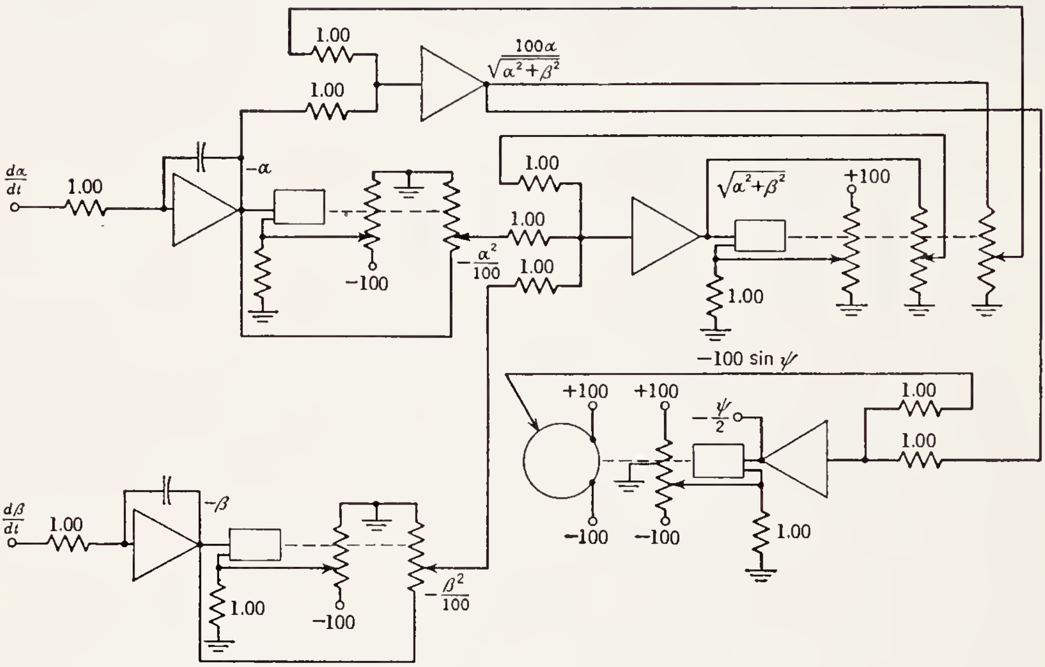

FIG. 6-21. Circuit diagram for solution of $\psi = \arcsin (\alpha/\sqrt{\alpha^2 + \beta^2})$.

Example. As an example of a problem involving the calculation of the inverse-sine function, let us consider the problems where we are given $d\alpha/dt$ and $d\beta/dt$ and are to compute ψ, where

$$\sin \psi = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \tag{6-23}$$

The range of the variables is

$$0 \leq \alpha < 100 \qquad 0 \leq \beta < 100$$

We shall program this equation using servomultipliers and resolvers. Since the resolver is scaled to produce the sine of the input voltage expressed in degrees divided by 2, our answer will be the angle $\psi/2$ expressed in degrees. Solving Eq. (6-16) for ψ, we obtain

$$\psi = \arcsin \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \tag{6-24}$$

To program this equation will require the use of three implicit-function circuits; these are square root, division, and arcsine. The square root and division can be accomplished on one servomultiplier if the output of the first high-gain amplifier is used to drive the servo. Figure 6-21 shows the final circuit for simulating Eq. (6-24). Programming Eq. (6-24) is reexamined as Prob. 7-5. Equation (6-24) is of interest, since it is exactly the form which arises when computing the angle of attack of an aircraft in terms of its component velocities.

6-6. THE ARCTANGENT CIRCUIT

Generally in most computer installations devices for generating tangent functions and inverse-tangent functions are not available (see Chap. 7 for the description of a suitable tangent-function generator). While the tangent function and its inverse are not available, we usually have on hand devices capable of generating sine and cosine functions. Suppose that we consider the relationship $e_i \cos e_o - \sin e_o = 0$. Solving for e_o, we obtain

$$e_o = \arctan e_i \tag{6-25}$$

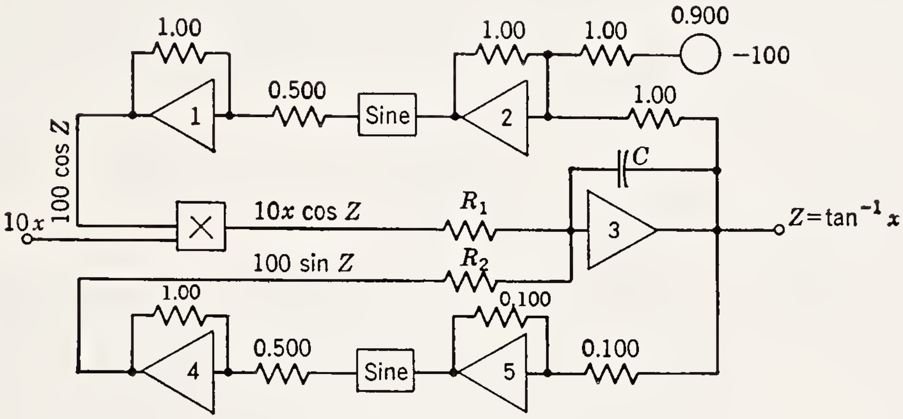

FIG. 6-22. The arctangent circuit.

Thus it is apparent that, if the output voltage of an amplifier is fed back through a device which generates the sine of the voltage e_o and also through a parallel path in which first the cosine of e_o is formed and then the product $e_i \cos e_o$ is obtained, the output voltage e_o will be proportional to the inverse tangent of the input voltage. Figure 6-22 shows a circuit for generating the arctangent. The sine and cosine of e_o are obtained by means of a special diode function generator described in Sec. 7-8. It is to be noticed that amplifier 3 has a small capacitor wired directly round it. Without this capacitor the circuit is ordinarily unstable for practical values of the resistors R_1 and R_2. Two approaches can be used to arrive at a stable circuit. These are to increase R_1 and R_2 to large values (15 to 20 megohms) or to use a small capacitor directly wired round the amplifier whose feedback is open. Both approaches result in limiting the frequency response of amplifier 3. Using a capacitor feedback with $R_1 = R_2 = R$, this circuit will be stable for $RC > 0.0001$. The effect of this time constant on the circuit results is negligible for ordinary computing frequencies.

If a servo resolver is used to form the sine and cosine functions, this instability will not ordinarily be encountered, the limited frequency response of the servo resolver acting as a filter. In this case, however, the response of the circuit is limited by the resolver response. The circuit of Fig. 6-22 will ordinarily have a wider bandwidth by a factor of 10 or more than the corresponding circuit using a servo resolver.

A nonimplicit circuit using a servo resolver and a servomultiplier can be used to generate a voltage proportional to the arctangent of another voltage. Considering the circuit of Fig. 6-23, let $e_i = 10 \tan \theta$ be the input voltage to a servomultiplier. If $\pm 100 \cos \theta$ is placed across a multiplying potentiometer of the servomultiplier, then the arm of this potentiometer will read $-10 \sin \theta$. Give $-10 \sin \theta$ a gain of -10, and apply it as input to a servo resolver. The resolver will rotate through an angle $\theta/2$ if the sine potentiometer, that is, $100 \sin \theta$, is used as feedback round the resolver loop. The cosine tap of the sine-cosine potentiometer furnishes $100 \cos \theta$ to put across the multiplying potentiometer of the servomultiplier. The voltage proportional to the arctangent is read off a linear potentiometer, with ± 100 volts across it, of the servo resolver.

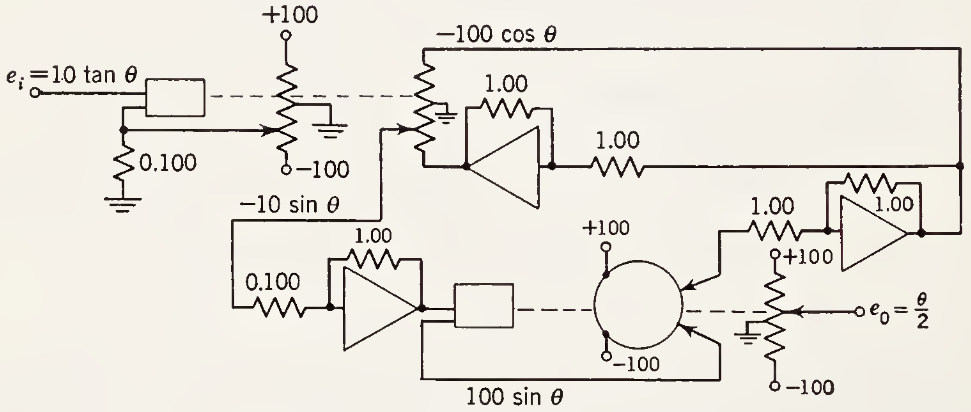

FIG. 6-23. Nonimplicit arctangent circuit.

Example. The differential equation

$$\frac{dy}{dt} + ay|\arctan y| = f(t) \tag{6-26}$$

involves the calculation of the arctangent. To program Eq. (6-26), we shall use the nonimplicit arctangent circuit of Fig. 6-23. Using this method for calculating the arctangent, the circuit of Fig. 6-24 results.

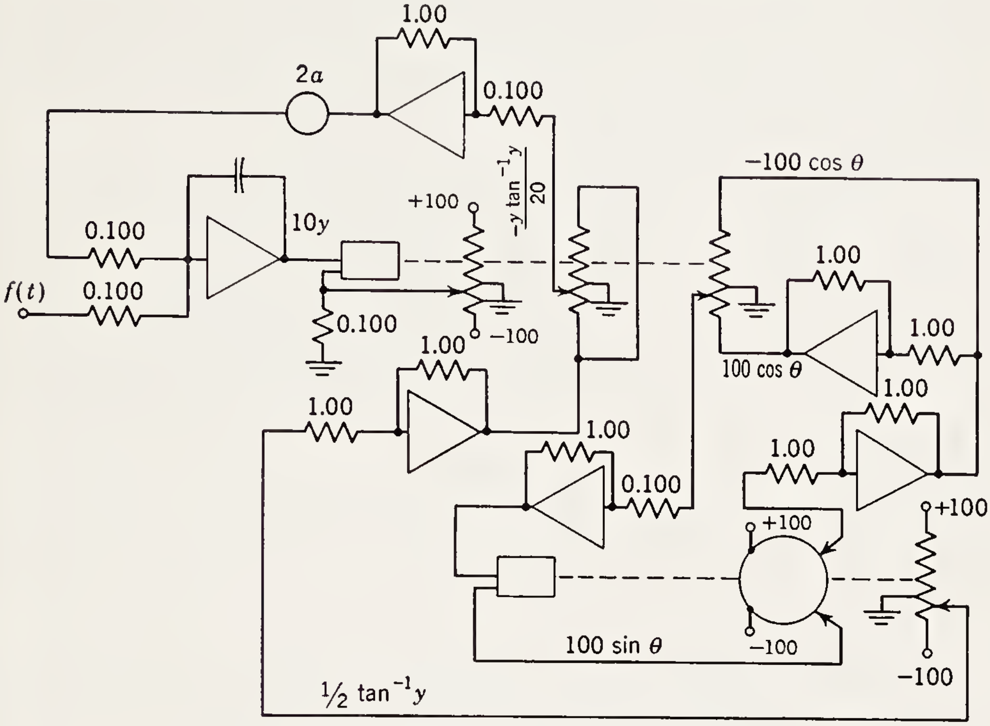

Fig. 6-24. Circuit for solving Eq. (6-26).

6-7. STABILITY OF IMPLICIT-FUNCTION CIRCUITS

The d-c operational amplifier is for most computing purposes considered to be a device which is independent of frequency. Thus the assumption is made that the d-c operational amplifier does not have dynamic characteristics. Ordinarily this assumption does not lead to any difficulty, as the dynamic characteristics of the amplifier are higher (by a factor of 10 or more) than those of the other computing components. In the case, however, where a series of amplifiers are connected together to form an algebraic loop (i.e., a loop containing no integration) the frequency-response characteristics of the amplifiers can produce unstable oscillations in the loop. In the case of implicit-function circuits the possibility of instability is particularly great. In particular, if the devices used to generate the inverse functions employed in the feedback have frequency-response characteristics comparable with those of the amplifier, unstable oscillations may result.

According to the Nyquist criteria* the condition for stability is that the gain round the loop must drop below unity at a frequency less than that which induces an accumulative 180-deg phase shift round the loop. In a given loop, if the amplifier frequency-response characteristics are combined to give the over-all characteristic, in general the results will show that the composite amplitude characteristic differs but slightly from that of a single amplifier (over the frequency range of interest), while the phase shift round the loop is considerable. Therefore, the conditions for instability can be easily realized.

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* See Refs. 2 and 3.

It may happen that it is desired to program a circuit in a manner which results in unstable oscillations. Also, it may not be easy to rearrange the circuit to avoid the oscillations. In this case a stable circuit may sometimes be realized by compromising the circuit frequency response. This is accomplished by making one or more of the amplifiers in the loop into a low-pass filter by introducing a small capacitor in parallel with the feedback resistor. For example, consider the normal summing amplifier whose feedback resistance is 1 megohm. If we parallel this with a 0.01-μf capacitor, we have $RC = 1 \times 0.01 = 0.01$ sec. This amplifier would then represent a first-order system with a break frequency of 15.9 cps. In most cases this would still leave a passband for computing which is acceptable.

In some computers a roll-off network is incorporated into the summing junction. This usually consists of a 1-megohm resistor in series with the input grid and in some cases a small capacitor to ground. The roll-off network affects the high-frequency stability characteristics of the amplifier. Some computers make use of the roll-off network when using the amplifier as an integrator but do not use it when using the amplifier as a summer. When using the amplifier in the high-gain mode, one will probably encounter fewer stability problems if the roll-off network is used.

The technique of using a small capacitance in parallel with the normal feedback resistance, of a summing amplifier, is generally applicable for overcoming high-frequency instability. Some computing amplifiers will not permit the use of this device unless a suitable roll-off network is used. Caution should be observed that the value of the capacitor is not so large as to introduce too much unwanted phase lag. The minimum time constant required for stability of a given circuit must be determined by experimentation.

6-8. ANALYSIS OF AN INDETERMINANT CASE

It may happen that a division circuit may have the divisor take on the value zero at $t = 0$ and be a nonzero voltage at all other times. This would mean that the division circuit would overload at the initial instant and be well behaved at all other times. Ordinarily this would mean that no solution could be obtained. However, an artifice can be employed which gets round this difficulty. Consider the function $y = Ae^{-\alpha t}$. This function has a magnitude of A volts at time $t = 0$ and decays to a value of A/e in $1/\alpha$ sec. Thus, if the function y is added to the divisor, the sum will be a finite voltage at $t = 0$ and will decay to the true divisor as time increases. Usually the value of A can be adjusted to be just enough to prevent overload, and the value of α just large enough so as to provide a voltage long enough for the true divisor to build up. The function $y = Ae^{-\alpha t}$ can be realized, using one amplifier as shown in Fig. 6-25. The value of A is determined by the initial condition on the integrator, and the value of α is equal to the product RC.

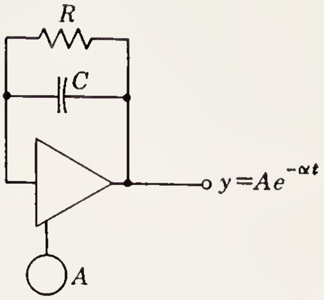

FIG. 6-25. Circuit for generating $y = Ae^{-\alpha t}$.

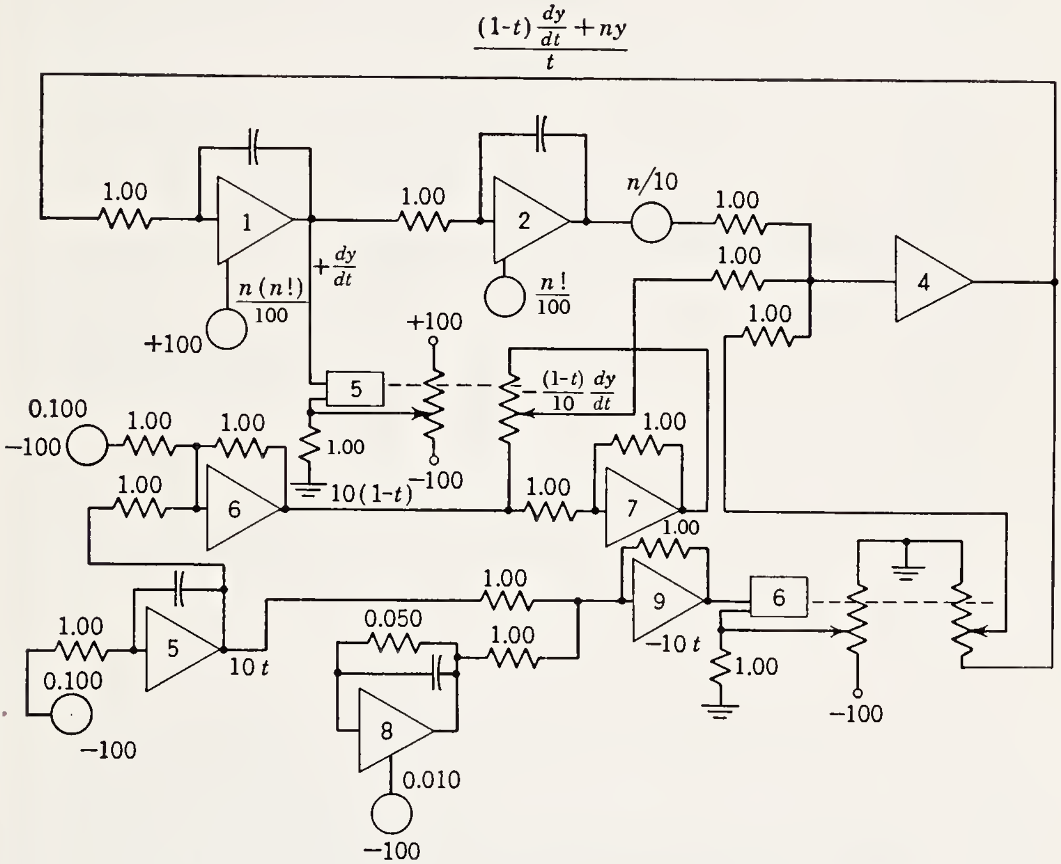

FIG. 6-26. Circuit for solution of Laguerre's differential equation.

Example. The differential equation, Eq. (6-27), known as Laguerre's differential equation

$$t\frac{d^2y}{dt^2} + (1 - t)\frac{dy}{dt} + ny = 0 \qquad (6\text{-}27)$$

is of considerable importance in mathematical physics. One solution to this equation consists of a series of orthogonal polynomials $L_n(x)$. The initial conditions are $y(0) = n!$, $y'(0) = n(n!)$. To program this equation for solution on the analog computer, we first divide by t and solve for the highest derivative, to obtain

$$\frac{d^2y}{dt^2} = -\frac{(1-t)}{t}\frac{dy}{dt} - \frac{ny}{t} \tag{6-28}$$

This equation as it stands will involve division by zero at $t = 0$. To avoid this difficulty, the technique given above is employed. In the circuit diagram of Fig. 6-26 amplifier 8 is used to furnish an initial voltage for the division circuit. In this application an initial condition of 1 volt applied so as to decay with a time constant of 0.05 sec is sufficient to prevent overload of the division circuit. The programming of Eq. (6-28) is carried out in a straightforward manner, the division circuit being performed by means of amplifier 4 and servomultiplier 6. Amplifier 5 is used as an integrator to obtain a voltage proportional to the time t.

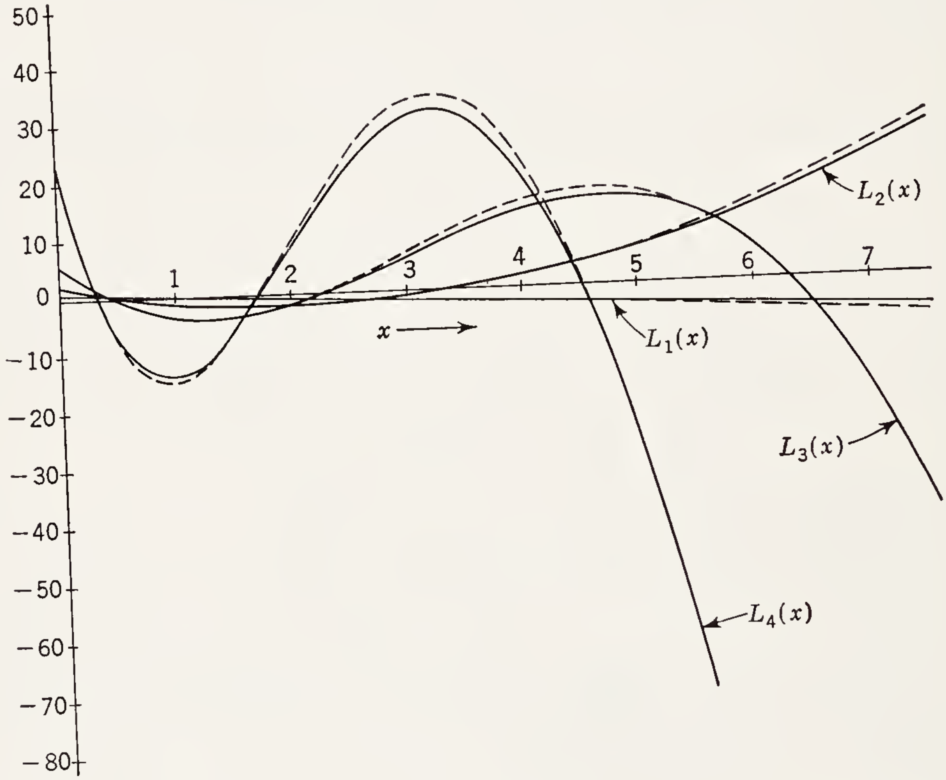

Fig. 6-27. Comparison of exact and analog solutions for Laguerre polynomials.

Figure 6-27 illustrates the magnitude of the error resulting from the introduction of the exponentially decaying voltage in the division circuit at $t = 0$. In this figure the analog-computer solution is compared with the exact known solution.

PROBLEMS

6-1. The simulation of the function $y = \log x$ is obtained by forming the differential equation whose solution is $y = \log x$ (see Sec. 2-12). When this is done there is obtained

$$y = \int_0^t \frac{1}{x}\frac{dx}{dt}\,dt$$

Assuming that dx/dt is available, diagram a circuit for the generation of $y = \log x$.

6-2. Legendre's differential equation is

$$(1 - t^2)\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + n(n+1)y = 0$$

where $-1 \leq t \leq 1$ and the initial conditions are

$$\left.\begin{aligned} \dot{y}(-1) &= 0 \\ y(-1) &= 1 \end{aligned}\right\} \quad \text{for } n = 0$$

$$\left.\begin{aligned} \dot{y}(-1) &= 1 \\ y(-1) &= 1 \end{aligned}\right\} \quad \text{for } n = 1$$

$$\left.\begin{aligned} \dot{y}(-1) &= -3 \\ y(-1) &= 1 \end{aligned}\right\} \quad \text{for } n = 2$$

$$\left.\begin{aligned} \dot{y}(-1) &= 6 \\ y(-1) &= -1 \end{aligned}\right\} \quad \text{for } n = 3$$

Program this equation to obtain solutions for $t = -1$ to $t = 1$, and make provisions in the circuit for n to have the values $n = 0, 1, 2, 3$.

6-3. Bessel's differential equation is

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - n^2)y = 0$$

where $0 \leq t < \infty$, and the initial conditions are

$$\left.\begin{aligned} \dot{y}(0) &= 0 \\ y(0) &= 1 \end{aligned}\right\} \quad \text{for } n = 0$$

$$\left.\begin{aligned} \dot{y}(0) &= 1 \\ y(0) &= 0 \end{aligned}\right\} \quad \text{for } n = 1$$

$$\left.\begin{aligned} \dot{y}(0) &= 0 \\ y(0) &= 0 \end{aligned}\right\} \quad \text{for } n = 2$$

Program this equation to obtain solutions for $t = 0$ to $t = 10$. Make provisions for $n = 0, 1, 2$.

6-4. The frequency-response characteristics of a second-order damped oscillator can be written in the form

$$\left|\frac{E_0}{E_i}\right| = \frac{1}{\sqrt{(1 - u^2)^2 + 4\xi^2u^2}}$$

$$\phi = -\arctan\frac{2\xi u}{1 - u^2}$$

where $u = \omega/\omega_n$ = nondimensional frequency
ξ = damping ratio — (damping present)/(critical damping)

Obtain a voltage u by integrating a constant, and establish a circuit for determining the frequency response for the range $u = 0$ to $u = 5$. Let $\xi = 0.05, 0.10, 0.20, 0.30, 0.40, 0.50, 0.60, 0.70, 0.80, 0.90, 1.00$.

6-5. The equation for the stability of an artillery rocket is

$$\sqrt{t}\,\frac{d^2\alpha}{dt^2} + \frac{1}{\sqrt{t}}\,\frac{d\alpha}{dt} + \left(\frac{4\pi^2\sqrt{t}}{\sigma^2} - \frac{1}{4t\sqrt{t}}\right)\alpha = \frac{\delta}{2k^2\sqrt{t}}\,\frac{4\pi^2\sqrt{t}}{\sigma^2}\,\gamma - \frac{1}{4t\sqrt{t}}\left(\beta + \frac{gm}{s}\cos\theta_i\right)$$

where α = angle of attack of rocket
s = thrust of rocket motor
β = angular misalignment of rocket motor
δ = moment arm of engine with respect to center of gravity
m = mass of rocket
g = acceleration of gravity
t = distance traveled from launching point
θ_i = inclination of the velocity vector
k = radius of gyration of the rocket about a lateral axis through the center of gravity
σ = characteristic length of disturbed motion of rocket
γ = angular misalignment of rocket fins

Prepare a computer diagram for this equation for the case of zero initial conditions.

6-6. The fuel flow through a metering orifice is given by

$$W_f = CA\sqrt{\Delta P}$$

where C = orifice coefficient = a constant
A = metering area
P = pressure differential across orifice

Given that the metering area varies as $A = b - x^2$, where x = displacement of metering orifice, set up a circuit for determining the pressure drop across the orifice as a function of the fuel flow and the displacement of the metering orifice.

6-7. Given velocity components U, V, W, so scaled as to have maximum values less than 100 volts. Using servomultipliers, set a circuit for computing the total velocity U_T.

$$U_T = \sqrt{U^2 + W^2 + V^2}$$

6-8. Considering Eq. (6-14), we can write $(k^2 + a^2\cos^2\theta)(d^2\theta/dt^2) - a^2\sin\theta\cos\theta\,(d\theta/dt)^2 = -ag\cos\theta$, and, on multiplying both sides by $2(d\theta/dt)$, the left-hand side becomes the derivative of a certain function with respect to t, so that the equation takes the form

$$\frac{d}{dt}\left[(k^2 + a^2\cos^2\theta)\left(\frac{d\theta}{dt}\right)^2\right] = -2ag\cos\theta\,\frac{d\theta}{dt}$$

On integrating each side of this equation with respect to t, we find

$$(k^2 + a^2\cos^2\theta)\left(\frac{d\theta}{dt}\right)^2 = -2ag\sin\theta + C$$

To determine C, make use of the initial conditions $\dot{\theta} = 0$ and $\theta = \alpha$, to get

$$(k^2 + a^2 \cos^2 \theta) \left(\frac{d\theta}{dt}\right)^2 = 2ag\ (\sin \alpha - \sin \theta)$$

Program this equation to obtain θ as a function of time.

6-9. The equation of motion of a spherical pendulum is

$$\frac{d^2\theta}{dt^2} = \frac{h^2 \cos \theta}{\sin^3 \theta} + \frac{g}{a} \sin \theta$$

where θ = colatitude
h, a, g = constants

A first integral can be obtained by multiplying by $2(d\theta/dt)$ to get

$$2 \frac{d\theta}{dt} \frac{d^2\theta}{dt^2} = \frac{2h^2 \cos \theta}{\sin^3 \theta} \frac{d\theta}{dt} + \frac{2g}{a} \sin \theta \frac{d\theta}{dt}$$

Integrating,

$$\left(\frac{d\theta}{dt}\right)^2 = - \frac{h^2}{\sin^2 \theta} - \frac{2g}{a} \cos \theta + k$$

Program this equation to obtain a solution for θ. Let the range of the constants and variables be $0 < \theta < \pi$ radians, $g = 32.2$ ft/sec, $a = 10$ ft, $h = 20$/sec.

6-10. Suppose the axle P of a gyroscope is caused to move in a smooth slot in the form of a meridian circle, which is made to rotate in any manner. The force F will then be normal to the meridian, or tangent to the parallel of latitude. Let the components of F along the meridian in the sense of the increasing θ and along the parallel of latitude in the sense of the increasing ψ be denoted, respectively, by Θ and Ψ. Then the equations of motion are

$$A \frac{d^2\theta}{dt^2} + \left(Cv - A \cos \theta \frac{d\psi}{dt}\right) \sin \theta \frac{d\psi}{dt} = \Theta$$

$$A \left(\sin \theta \frac{d^2\psi}{dt^2} + 2 \cos \theta \frac{d\theta}{dt} \frac{d\psi}{dt}\right) - Cv \frac{d\theta}{dt} = \Psi$$

where v = constant spin velocity of gyroscope, radians/sec
A = moment of inertia normal and tangent to spin axis
C = moment of inertia about spin axis

If Θ and Ψ are known functions of θ, ψ, t, these equations suffice to determine the path of P. For the case where

$v = 25{,}000$ radians/sec
$A = 200$ g cm^2
$C = 320$ g cm^2
$\Theta = \Psi = at(1 - e^{-3t})$

set up a circuit for solution for θ and ψ. Let the ranges of θ and ψ be 0 to 90 deg.

6-11. Solve the differential equation

$$\frac{d^2y}{dt^2} + 2ay \left[1 + \left(\frac{dy}{dt}\right)^2\right]^{1/2} = 0$$

with a an arbitrary constant and zero initial conditions.

6-12. A differential equation of the type

$$t^4 \frac{d^2\theta}{dt^2} + \theta^\mu = 0$$

arises in connection with the gravitational equilibrium of a gaseous configuration in stellar structure. Program this equation so as to obtain solutions for $\mu = 2, 3, 4$.

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CHAPTER 7

Arbitrary-function Generators

7-1. INTRODUCTION

Frequently the problem arises where it is necessary to generate arbitrary functions of an independent variable. Several methods for arriving at a satisfactory approximation to arbitrary functions exist. Most of these methods use either curve-following devices or line-segment approximations to the arbitrary function. The curve-follower type has been used in the past to a greater extent than any other. The curve-follower function generator can, under favorable conditions, reproduce a given function to approximately 1 per cent. In most cases the curve-follower arbitrary-function generator is difficult to set up.

The line-segment function generator creates a sequence of straight or curved lines which are connected together to form the desired function. Many of these function generators have some type of switching element such as a diode or relay. If diodes are used, there are no moving parts and the frequency response of the function generator is as good as that of the amplifiers used in conjunction with the circuit. Present-day usage of arbitrary-function generators shows an increasing dependence on line-segment methods.

In the class of arbitrary-function generators there exist special-purpose function generators which are developed and programmed to generate common elementary functions such as sin x, arctan x, x^2, etc. These functions can be generated using the general-purpose function generators, but since they occur so frequently, it is economical to prepare a small, inexpensive function generator which generates only a particular function.

Current activity in the field of arbitrary-function generators is centered around the development of devices capable of generating functions of two variables. The devices used to accomplish this are mostly extensions of the older techniques used for functions of one variable. One exception is the recent development of a semiconducting sheet for the generation of a certain class of functions of two variables.*

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* Ref. 3.

7-2. CURVE-FOLLOWER FUNCTION GENERATOR USING RESISTANCE ELEMENTS

A curve-following type of function generator can be constructed to be used in conjunction with a servomultiplier. The cam type developed by the Reeves Instrument Company is the most common of these units and is described in this section. The arbitrary function is cut as a groove on the surface of a cylindrical cam, of insulating material, approximately 4 in. long and 1½ in. in diameter. A wire is pressed into the groove, and one end of the wire is connected to a conducting ring which makes a contact with a brush. The cam is mounted in a cam holder in such a manner that the wire in the groove makes contact with a 20-kilohm resistance card across which 100 volts is impressed. As the cam is rotated by the independent variable, the wire picks off a voltage proportional to the desired arbitrary function. The rotation of the cam by the independent variable is usually accomplished by gearing the cam to multiplying potentiometers of a servomultiplier. In some applications a special servomechanism can be constructed for driving a series of cams. This technique has been applied by the authors to drive a chain consisting of 20 cams by one servomechanism. The total assembly of the function generator consisted of the 20 cams, for generating 20 arbitrary functions of the angle of attack of an aircraft, connected end to end and driven by a common servomechanism at one end. Some difficulty was experienced with twisting of the driving shaft since the total assembly consisted of some 18 ft of ⅜-in. steel rod coupled together in approximately 10-in. lengths. By going to a solid coupling the differential angular displacement due to shaft twisting from one end to the other was less than 2 deg. This type of function generator suffers from the disadvantage of the wire making sliding contact with the resistance card, and frequent mechanical failure of these elements was experienced. The frequency response of this element was also quite poor (flat to approximately 2 cps). It should be noted that in spite of the difficulties mentioned the cam-type curve follower did give satisfactory solutions although considerable patience was required during calibration and operation.

For the reason mentioned, and because the more modern servomultipliers do not permit coupling cams to the multiplying potentiometers, the cam-type curve follower seems destined to disappear.

Another variation of the same principle exists in an application of the input table. This is a device whereby arbitrary input functions of time or any problem variable are normally introduced into the computer by means of hand tracking. The arbitrary function is plotted on an ordinary piece of graph paper and attached to a cylindrical drum whose rotation is controlled by a servomechanism. If a piece of wire is glued to the graph and a suitable resistance card fitted to the proximity of the cylindrical drum, then the equivalent of the cam-type curve follower is realized.

7-3. PHOTOFORMER-TYPE CURVE FOLLOWING FUNCTION GENERATOR

A second type of curve-following function generator consists of a cathode-ray tube in front of which is placed a mask cut in the shape of the function to be generated. A photocell is placed so that it looks at the contour of the mask. The general arrangement of the circuit is shown in Fig. 7-1, and its operation can be explained in the following manner:

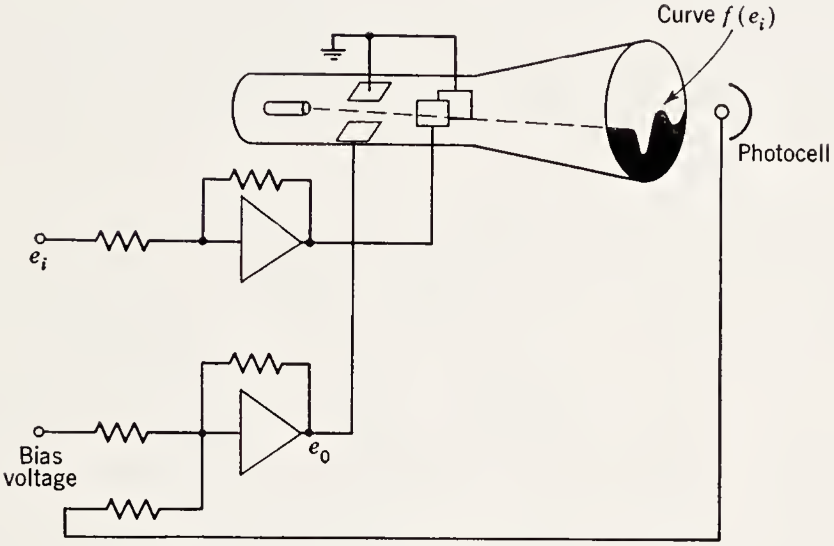

FIG. 7-1. Functional diagram of a photoformer.

The input voltage e_i is applied between the horizontal-deflection plates of a cathode-ray tube through a suitable d-c amplifier. The voltage e_o between the vertical-deflection plates is made to be proportional to the arbitrary function to be generated. Thus

$$e_o = f(e_i) \tag{7-1}$$

i.e., the output voltage e_o is made to vary as a function of the input voltage e_i. This is accomplished by a feedback arrangement which forces the electron beam to ride on the edge of an opaque mask placed over the lower portion of the cathode-ray screen.

The feedback loop which forces the beam to ride on the edge of the mask consists of bias voltage which tends to force the electron beam upward away from the mask and a photocell mounted in front of the screen which applies an error voltage across the input terminals of the vertical-deflection amplifier. This error voltage is phased so that the beam is forced downward. The two voltages are equal and opposite when the beam rides on the edge of the mask. Thus, as the input voltage e_i varies, the output voltage generates the required arbitrary function $e_o = f(e_i)$ if the mask has been made to have the proper shape.

A special photographic process has been developed for preparing the mask. This procedure makes it necessary only to plot the curve of the arbitrary function on a special paper. A photograph is then taken, and the mask appears as a slide.

The photoformer can be made to give good accuracy and has excellent frequency response. The calibration of the slides can be difficult. During operation the circuits tend to drift sufficiently so that frequent adjustment is required.

7-4. TAPPED-POTENTIOMETER FUNCTION GENERATORS

The servomultiplier can be made to serve as a function generator. This is accomplished, as indicated in Fig. 7-2, by tapping the multiplying potentiometers at equally spaced intervals throughout their operating range. A voltage is connected to each of the taps from external calibration potentiometers. The wiper of the tapped multiplying potentiometer is driven by the input voltage e_i, and the output voltage varies according to the voltages set at each tap. In this way, if there are N taps, the function is represented by as many as $N - 1$ line segments.

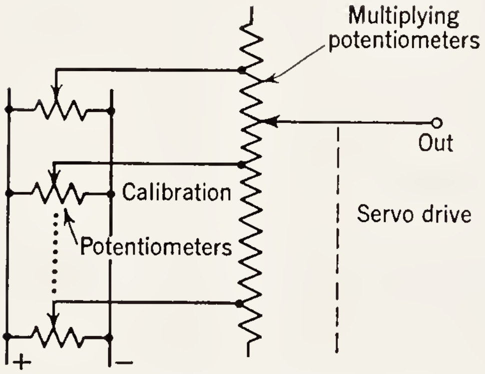

FIG. 7-2. Tapped potentiometer with loading circuit.

A method of calibrating tapped potentiometers which is particularly simple in operation has been developed by Electronics Associates, Inc. In this particular function generator each tapped potentiometer has 17 taps and therefore represents a function of 16 line segments. To each tap of the tapped potentiometer is connected a padding potentiometer controlled by a switch which can connect the top of the potentiometer to the positive or negative reference voltage or to ground or leave it open. By varying the resistance in series with each tap, it is therefore possible to provide any desired voltage at each tap.

In adjusting the function generator, the desired voltages for each tap are first set up on precision potentiometers provided on the control panel. The particular tap being adjusted is then compared, by means of a null meter, with the voltage previously set on the corresponding precision potentiometer.

To prevent interaction of the taps, the tap voltages are adjusted in sequence, starting with the first tap. As indicated in Fig. 7-3, the next tap in sequence is fed its true tap voltage during the adjustments of the preceding tap. This is accomplished by feeding the voltage from the precision potentiometer through an isolation amplifier of unity gain.

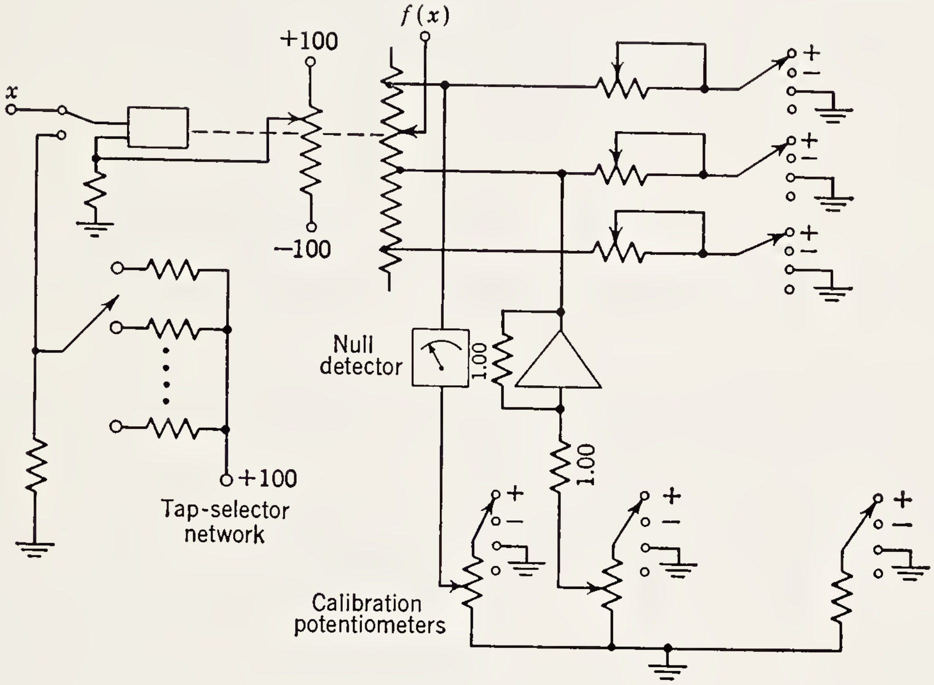

Fig. 7-3. Electronics Associates scheme for setting tapped potentiometers.

Since the isolation amplifier acts as a regulated voltage source, the tap that it feeds is forcibly held at the proper voltage. This in turn provides the proper loading on the tap being set.

A multiposition switch is provided that allows the convenient selection of the tap that is to be set and also positions the servo so that the wiper of the tapped potentiometer is positioned to the tap that is being set.

The tapped potentiometer can be made to simultaneously generate an arbitrary function $f(e_1)$ and multiply this function by a second voltage e_2 to obtain $f(e_1)e_2/100$. This feature can be of considerable importance in the simulation of aerodynamic forces and moments. In this case these forces and moments usually appear in the typical form $C_{m\delta_e}\delta_e$, where $C_{m\delta_e}$ is the pitching moment due to elevator deflection δ_e. Now, in general, when the elevator is deflected, there will be generated forces and moments on the airplane about all axes. These forces and moments in general will be arbitrary functions of Mach number, angle of attack, and/or angle of sideslip. In most cases the behavior of the aerodynamic forces as a function of but one single paramcter will be of interest. Thus, the simultaneous generation of several aerodynamic forces and moments and the multiplication of these by the elevator deflection are called for. The tapped potentiometer can be made to fit this requirement admirably well. Figure 7-4 shows a circuit for generating the moments and forces on a VTOL-type aircraft due to elevator deflection and simultaneously multiplying these quantities by the elevator deflection. The moments and forces are arbitrary functions of angle of attack α.

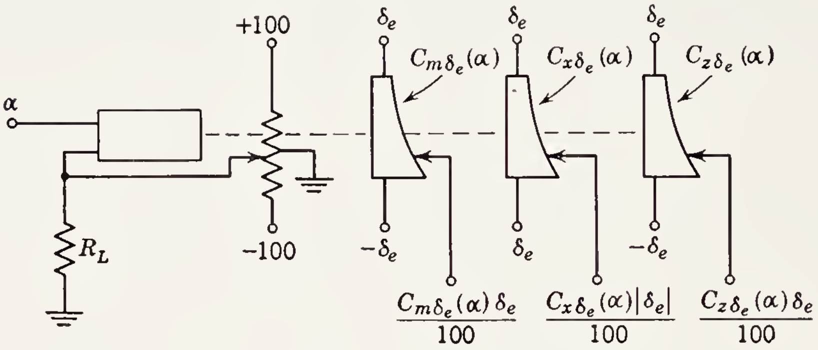

FIG. 7-4. Function generation simultaneous with multiplication, using tapped potentiometers.

In Fig. 7-4 the symbol given in the table of Fig. 2-1 has been used to indicate using the tapped potentiometer as a function generator. To perform multiplication simultaneously with function generation, the functions are first set up with ±100 volts reference. After calibration the ±100-volt references are replaced by the variable to be multiplied by.

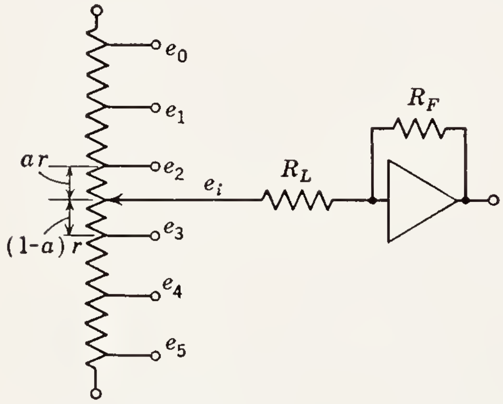

FIG. 7-5. Tapped potentiometer with load.

To investigate the effect the load on the wiper arm has on the assumed linear interpolation between taps, we consider Fig. 7-5. In this figure, if r is the incremental resistance between taps, a the fraction of the distance between taps transversed, and R_L the load resistance, then the nodal equation is

$$\frac{e_2 - e_i}{ar} + \frac{e_3 - e_i}{(1 - a)r} - \frac{e_i}{R_L} = 0 \tag{7-2}$$

Upon solving Eq. (7-2) for e_i, there is obtained

$$e_i = \frac{1}{1 + (1 - a)ar/R_L}[(1 - a)e_2 + ae_3] \tag{7-3}$$

The term $(1 - a)ar/R_L$ causes the interpolation between taps to be nonlinear. In most practical designs $R_L = 1$ megohm,

$$r = 30 \text{ kilohms}/17 = 1.76 \text{ kilohms}$$

and the effect of this term is a small fraction of 1 per cent.

7-5. DIODE FUNCTION GENERATORS

Diode function generators come in an almost endless variety of circuits. Most of the circuits utilize the diode to act as a high-speed voltage-sensitive switch. The operation of three of the more common commercial units is described. The first circuit described forms the basis for the GEDA diode function generator, model N3E. The second circuit described forms the basis for the EASE model 1070 function generator. The third circuit described forms the basis of the REAC function generator, model FG 400.

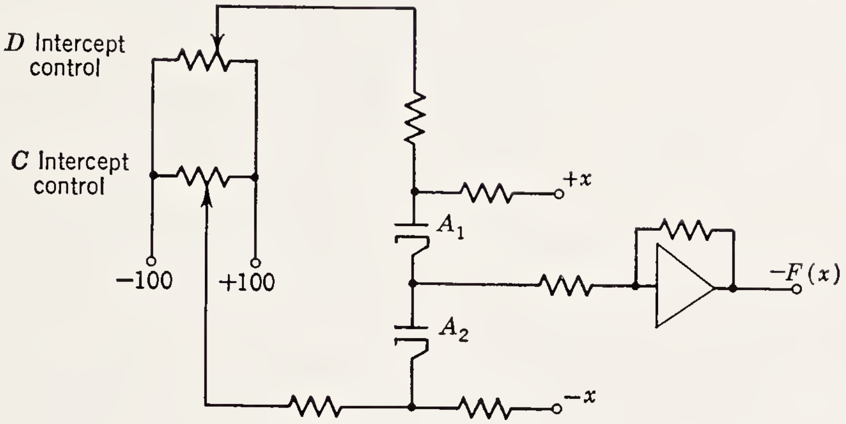

Fig. 7-6. Basic diode pair for GEDA diode function generator.

Consider a circuit consisting of a pair of biased diodes wired so that their outputs are opposed. This pair of diodes (Fig. 7-6) act as voltage-sensitive switches for introducing the input variable into the operational summing amplifier. The biasing circuit, provided for each diode pair, permits either of the single diode sections to conduct individually over any portion of the entire input-voltage range.

The operation of the circuit of Fig. 7-6 can be explained as follows: The $+x$ input is applied through a resistor to the plate of diode A_1, and the $-x$ input is applied through a similar resistor to the cathode of diode A_2. Both polarities of the x input must always be supplied. To avoid the effects of loading, the $\pm x$ inputs must be supplied from a low-impedance source such as an amplifier, even though meeting this requirement may entail the use of an additional amplifier in the problem setup. If the negatively sloped line segment shown in Fig. 7-7 is to be generated, diode A_1 must be made to conduct first.

When A_1 conducts, the current flow results in a positive voltage at the input of the summing amplifier and, consequently, a negative voltage at the amplifier output. The point of conduction of A_1 is determined by the setting of the D intercept control. Thus, if a positive voltage β is applied to the plate of the diode A_1 by the D intercept control, the diode A_1 will conduct until the $+x$ input becomes more negative than β.

The line segment is terminated by conduction of diode A_2. Since A_2 is wired into the circuit in reverse of A_1, it will produce a positive amplifier output. To terminate the line segment as shown in Fig. 7-7, A_2 is made to conduct at values of x greater than $+\alpha$ volts by presetting the C intercept control to apply $-\alpha$ volts to its cathode. A_1 continues to conduct for input voltages more positive than α, but since it is opposed by an equal and opposite voltage from A_2, the resultant amplifier output is constant.

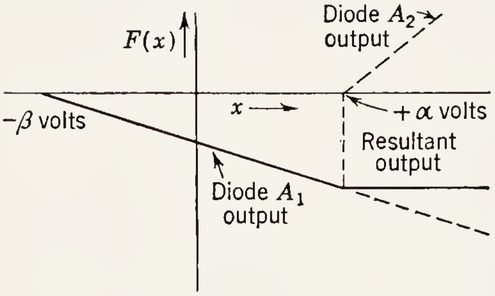

FIG. 7-7. Negatively sloped line segment.

Thus, in generating a negatively sloped line segment, diode A_1 must be made to conduct before A_2. For a positively sloped line segment, diode A_2 must be made to conduct before A_1. In the latter case the line segment must be initiated by a C intercept control and terminated by a D intercept control. If a series of these diode pairs as shown in Fig. 7-6 is used, then an arbitrary function can be approximated by a series of straight-line elements, as shown in Fig. 7-8. The characteristic of a function approximated by a combination of straight-line segments is determined by the slope of the individual lines and the intercepts, or values of input voltage at which the lines begin and end. An initial bias is applied to the output amplifier which sets the initial ordinate level of the entire function (see b, Fig. 7-8). The manner in which this constant output is modified over each incremental x region by the output of successive diode sections can also be noted in this figure. The slope direction and the value of the intercept X_1, $F(X_1)$; X_2, $F(X_2)$, etc., are determined by intercept controls. The degree of slope is determined by the gain of the output amplifier associated with a particular diode pair.

A somewhat different approach to the diode function generator is illustrated by the EASE model 1070. The basic diode circuit for this function generator is shown as Fig. 7-9. Each network receives the same input voltage x, and each has its positive output connected to the + terminal and its negative output connected to the − terminal. The

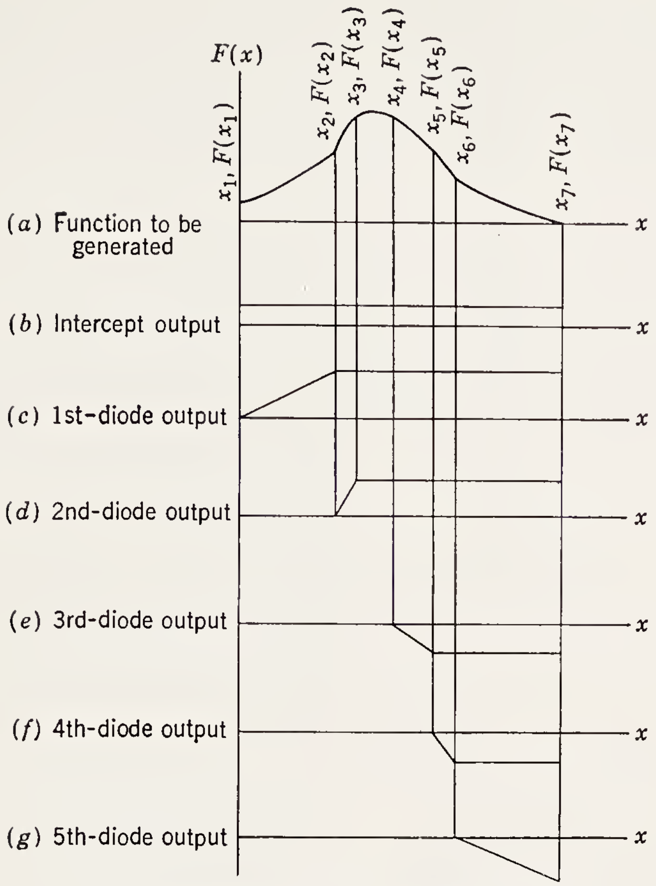

FIG. 7-8. Straight-line segment approximation.

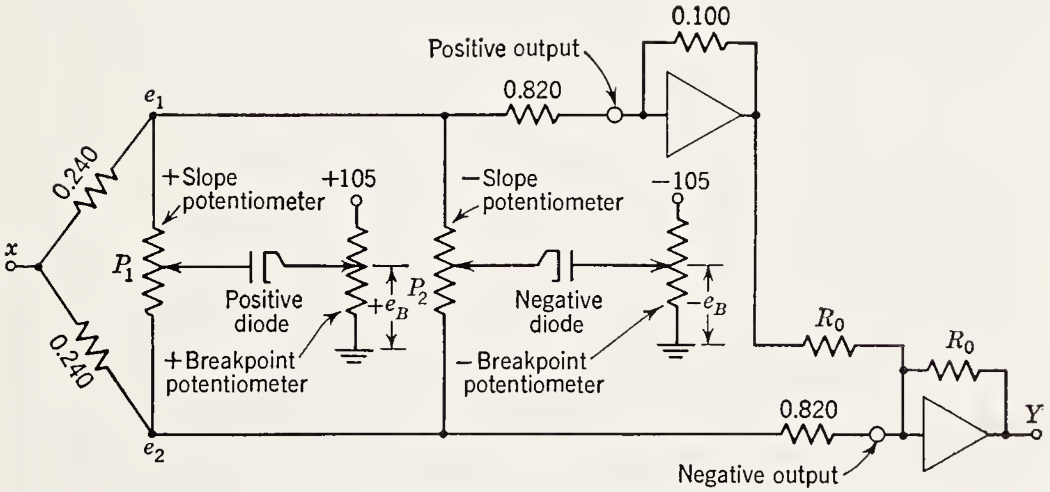

FIG. 7-9. Basic circuit for Beckman Instruments function generator.

output voltage y is proportional to the sum of all currents flowing to the $+$ terminal minus the sum of all currents flowing to the $-$ terminal. The contribution of each diode network to this current sum is the current flowing to its positive output minus the current flowing to its negative output. If we call this net current i_0, then

$$Y = K(i_{0_1} + i_{0_2} + i_{0_3} + \cdots + i_{0_n}) \tag{7-4}$$

Consider first the action of the individual diode network shown in Fig. 7-9, where $+e_B$ and $-e_B$ are the plus and minus break voltages. As long as x lies between these values, neither diode conducts. Owing to the symmetry of the circuit, $e_1 = e_2$, and the current at the positive output equals the current at the negative output. Then, $i_0 = 0$, and the network has no effect on Y. When x becomes more positive than e_B, the positive diode conducts. If P_1 is set at the mid-point, the circuit remains symmetrical and e_1 still equals e_2. If, however, P_1 is set so that the resistance from the diode to point e_1 is greater than that to the point e_2, e_1 will become more positive than e_2 as x rises. In this case, the current at the positive output exceeds that at the negative output, and i_0 is positive. If P_1 is set so that the resistance from the diode to point e_1 is less than that to point e_2, e_1 will become less positive than e_2 as x rises and i_0 will be negative. Stated quantitatively,

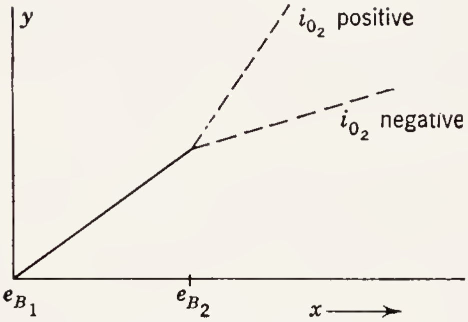

FIG. 7-10. Output characteristic of diode section.

$$e_1 - e_2 = a(x - e_B) \tag{7-5}$$

where a is a positive or negative constant determined by the setting of P_1 and

$$i_0 = k(e_1 - e_2) \tag{7-6}$$

where k is a constant determined by the 820-kilohm resistors. Therefore

$$i_0 = ka(x - e_B) = b(x - e_B) \tag{7-7}$$

where b is still a constant whose value and sign are adjustable by P_1.

To see how the action of each diode network affects Y, suppose that only the first positive diode with a break voltage of zero is conducting. The conducting network contributes a current i_{0_1} equal to b_1x so that Y increases, as diagrammed in Fig. 7-10, up to e_{B_2}. When x reaches e_{B_2}, the second positive diode begins to conduct, contributing i_{0_2} which also varies linearly with x. If the slope potentiometer is set so that i_{0_2} is positive, the slope of Y increases at this point. If i_{0_2} is made negative, the slope decreases. The effect of the second diode is described mathematically in the following equations: At all times

$$Y = K(i_{0_1} + i_{0_2} + \cdots) \tag{7-8}$$

From e_{B_2} to the next break point e_{B_3}

$$Y = K[b_1x + b_2(x - e_{B_2})] \tag{7-9}$$

Therefore

$$\Delta Y = K(b_1 + b_2)\,\Delta x \tag{7-10}$$

Note that b_2 (set by P_1) adds or subtracts from the previous slope coefficient b_1. As x reaches other succeeding break points, the slope of Y changes similarly. For example, from e_{B_3} to e_{B_4}

$$\Delta Y = K(b_1 + b_2 + b_3)\,\Delta x \tag{7-11}$$

Thus in this manner an arbitrary function is generated.

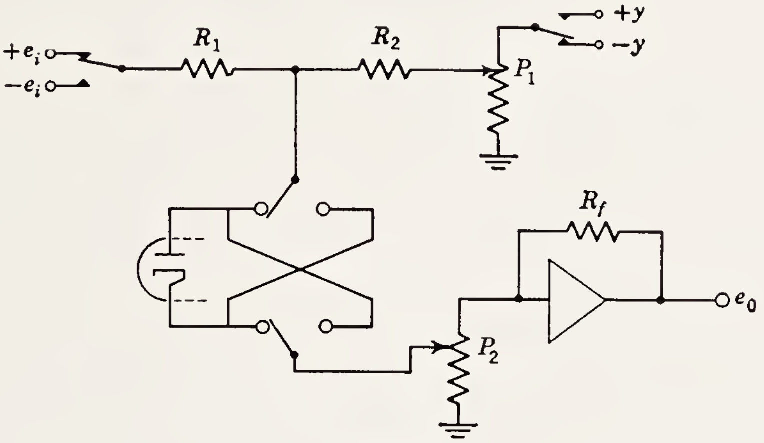

Fig. 7-11. Single diode channel of REAC function generator, model FG 400.

Figure 7-11 is a diagram of the diode circuit used as the basis of the REAC function generator, model FG 400. This particular function generator is of interest, as provisions are made whereby this unit can be used to generate certain classes of functions of two variables.

The application of this circuit to functions of a single variable is illustrated in Fig. 7-12, which shows the output voltage e_o of the amplifier as a function of the input voltage e_i. In the case of a function of a single variable the bias voltages $\pm y$ are supplied from a fixed reference source. In our case we shall assume this to be ±100 volts. Since the function to be generated has both positive and negative slopes, both signs of the input voltage are required. The initial slope and initial intercept are supplied by connecting both e_i and $+y$ directly into the output amplifier.

As indicated in Fig. 7-12, the output is a piecewise linear function consisting of segments C_0, C_1, C_2, . . . , C_8 which form an approximation to the given arbitrary function $f(e_i)$. The segments are joined together at the break points $B_1, B_2, \ldots, B_8$. The first segment represents the output of the circuit over a range of e_i, where none of the diodes are conducting. As e_i increases, the individual diode channels become conductive in successive steps. For the function generated in Fig. 7-12 diodes 4, 5, and 6 are connected to the $+e_i$ input, and diodes 1, 2, 3, 7, and 8 are connected to the $-e_i$ input. The corresponding $-y$ and $+y$ bias voltages are connected as required, respectively.

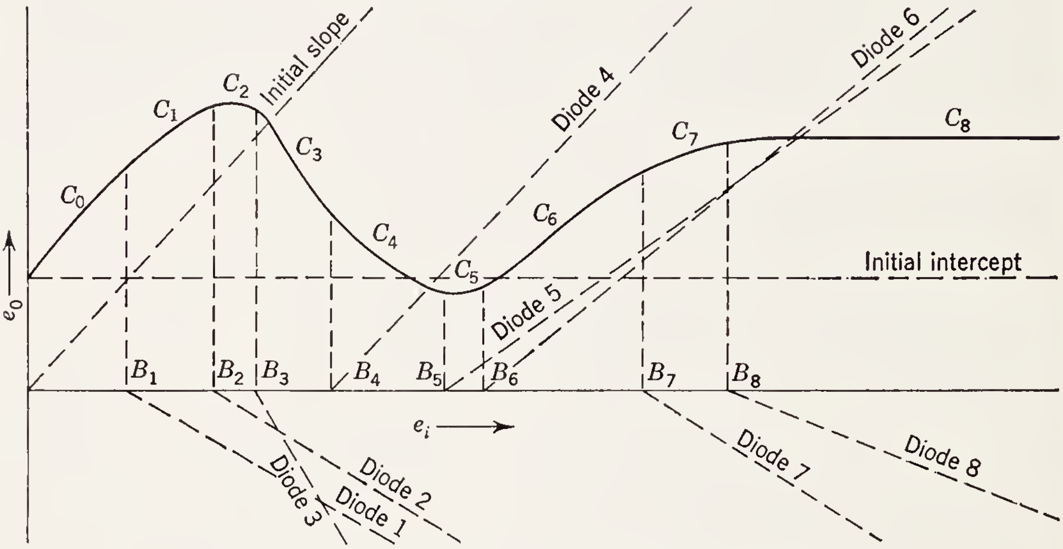

FIG. 7-12. Output of arbitrary-function generator.

The contributions of the individual diode channels to the total output voltage e_o are shown as the dashed lines in Fig. 7-12. These are broken lines containing a horizontal portion for $e_i \leq e_{B_i}$, where e_{B_i} is the break-point voltage of the ith diode and a ramp function of positive or negative slope, depending on the sign of e_i applied to the diode. The break-point voltage is determined by the setting of potentiometer P_1. The maximum slope is determined by the ratio of R_f/R_i, with potentiometer P_2 set all the way up. As the setting of potentiometer P_2 is decreased, the slope of the approximating ramp decreases.

7-6. SPECIAL-PURPOSE FUNCTION GENERATORS

The occurrence of some of the more elementary functions is so common in analog-computer work that it is convenient to build special-purpose function generators which can be adapted to these functions. The particular approach to the simulation of the elementary functions discussed in this section is not common practice throughout the field. The diode circuit used is applicable for the generation of any singlevalued monotonic mathematical curve but is particularly applicable to monotonic functions which are concave downward. Accuracy is theoretically limited only by the number of straight-line segments necessary to approximate the desired curve. In practice, component specifications fluctuate with temperature and age, thus limiting the maximum accuracy. Further, the fit to the desired curve is somewhat better than the straight-line approximation, as the diodes do not have a sharp cutoff characteristic. Rounding off of the corners tends to improve this straight-line fit to the arbitrary curve. Practical application of the circuit has achieved absolute accuracies as good as 0.25 per cent.

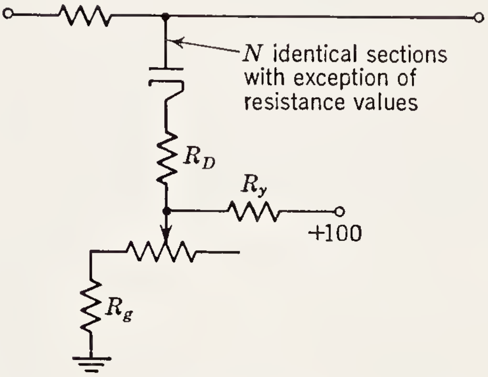

Fig. 7-13. Positive half of basic diode circuit for special-purpose function generators. Negative side is identical with diodes reversed.

Figure 7-13 shows the basic diode circuit round which the special-purpose function generators are built. It will be noticed that this circuit does not require an isolated power supply but can be packaged as a plug-in unit to operate from the reference ±100-volt power supply available in the analog computer.

By a proper selection of the values of the resistances R_y and R_g, the diode can be made to conduct at any desired positive or negative value (depending on the sign of the bias voltage) of the input voltage. When the first diode conducts, the resistor R_D acts as a voltage divider so that the voltage seen by an operational amplifier downstream of the function generator is a straight-line function of the input voltage. When the input voltage exceeds the voltage necessary for conduction of the second diode, this diode will also conduct. In doing so, it parallels a second resistor, with the original R_D placing the input of the computing amplifier closer to ground, thus reducing the slope of the approximating curve. The process continues, with more diodes coming into play as the input voltage increases, and the slope of the approximating curve continually decreases as more diodes are activated.

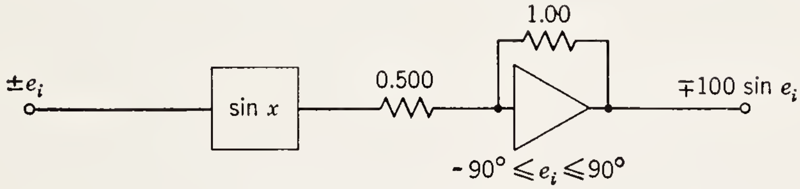

Fig. 7-14. sin *x* circuit.

Since sin x is a symmetrical monotonic curve and is concave downward, it lends itself naturally to the type of approximation described above. Figure 7-14 shows the circuit for sin x. Seven diodes will provide an absolute accuracy of ±0.25 per cent. The cosine function for the range $0 \leq e_i \leq 180$ is obtained by using the same diode circuit as the sine generator, with 90 deg $- x$ as the input voltage.

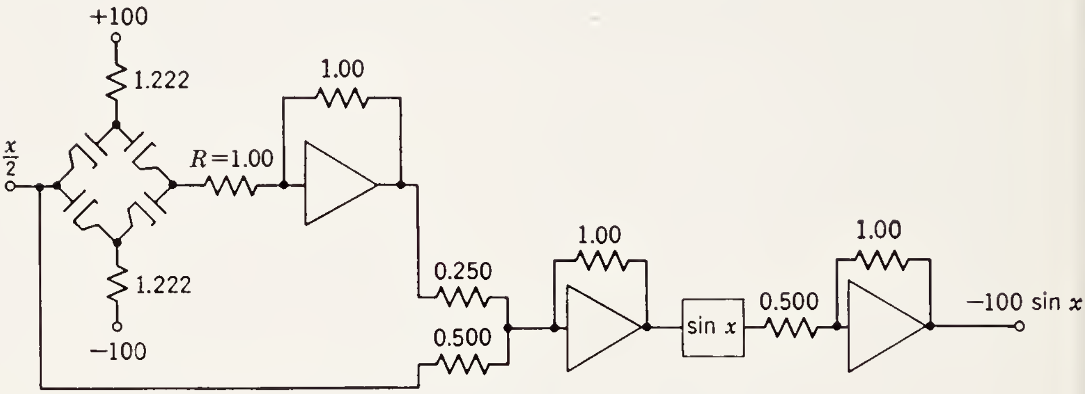

FIG. 7-15. Sine-function range-extender circuit.

The range of the sine-function generator can be simply extended to ±270 deg by the simple circuit shown in Fig. 7-15. The limiting circuit in front of the sine generator draws the curve shown as Fig. 7-16.

The inverse trigonometric functions arcsin x and arccos x are computed by the implicit-function technique explained in Chap. 6.

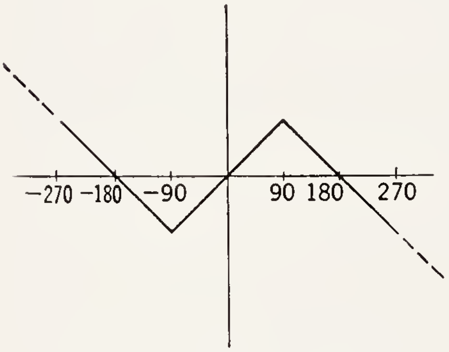

FIG. 7-16. Limiter output.

It was mentioned in Sec. 6-6 that a better technique existed for generation of the arctangent curve. In the case of the tangent family the function generator is set up to approximate the arctangent curve, since it is the only member of the family which meets the requirement of being a monotonic curve which is concave downward. If a scale factor of 10 is selected, then the circuit can be made to approximate the arctangent curve for $-87.3 < x < 87.3$. The tangent of an angle $x = \tan\theta$ is generated by the technique of implicit-function generation.

A circuit in which a diode function generator is used to perform four-quadrant multiplication is shown as Fig. 7-17. Four computing amplifiers and two diode units are required to accomplish multiplication. The diode multiplier units are packaged to have one input $\pm z$ and two outputs. When the output of the diode multiplier is loaded with an imped-

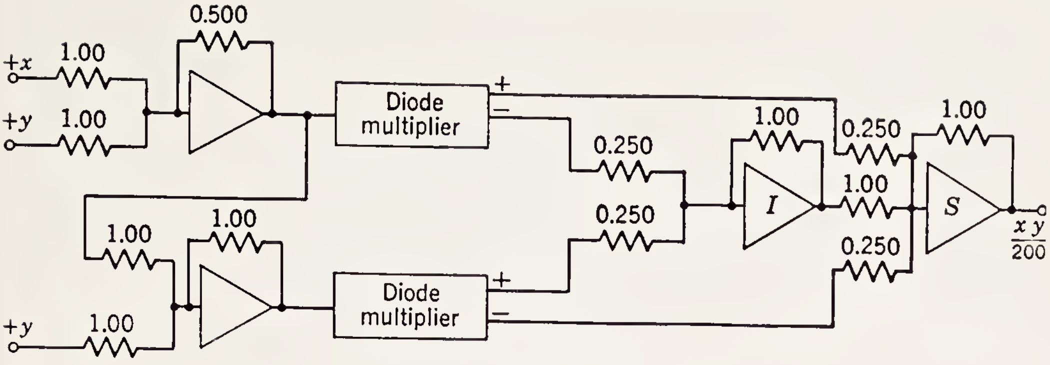

FIG. 7-17. Diode multiplying unit.

ance of 250 kilohms to ground, the output voltage approximates the curves

$$q_{(+)} = \frac{z - z^2/200}{4} \qquad z > 0$$

and

$$q_{(-)} = \frac{z + z^2/200}{4} \qquad z < 0$$

$$q_{(+)} = \frac{z}{4} \qquad z < 0 \tag{7-12}$$

and

$$q_{(-)} = \frac{z}{4} \qquad z > 0$$

These curves are shown graphically in Fig. 7-18.

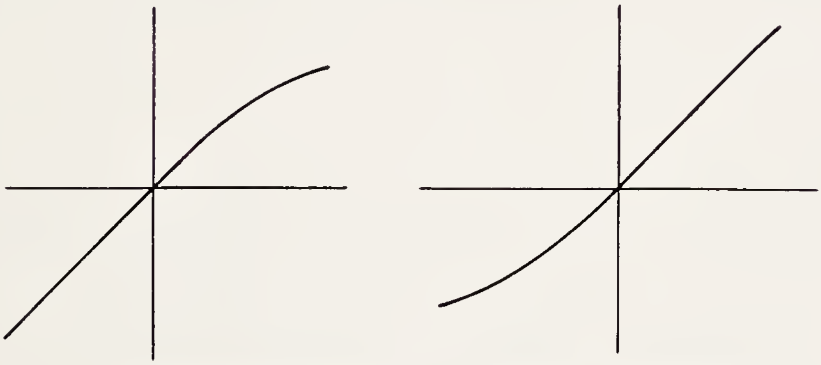

FIG. 7-18. Approximating curves for diode multiplier.

Use of the inverting amplifier I permits cancellation of the linear component of the nonlinear portion of the curve in the summing amplifier S, giving an output proportional to z^2. Applying two of the units as shown in Fig. 7-17 accomplishes multiplication in the conventional quarter-square manner.

For the case of the diode multiplier seven diode sections are used, allowing break points to be calculated so that an absolute accuracy of 0.513 per cent was achieved. Note that multiplication of small numbers is subject to a very small error, since in this region none of the diodes are conducting and the output of the amplifiers is entirely dependent upon the input resistors.

For the circuit shown as Fig. 7-17, $+x$ and $+y$ inputs give $xy/200$. However, it is interesting to note that, by interchanging + and − outputs from each diode unit, $+x$ and $+y$ inputs give $-xy/200$.

Division is accomplished by the conventional method of using the multiplying circuit as feedback to a high-gain amplifier.

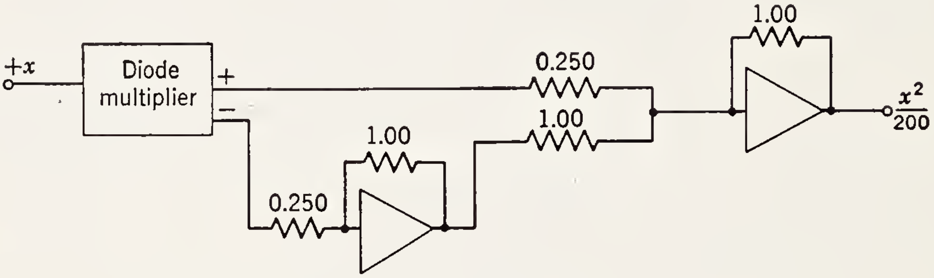

FIG. 7-19. Diode circuit for generating the square function.

The multiplier units can be used to perform the squaring operation. In this case only a part of the multiplier circuit is required. Figure 7-19 shows a circuit which can be used to give either $+x^2$ for $+x$ input or $-x^2$ for $+x$ input by simply changing the second amplifier to operate from the positive rather than the negative output of the diode generator.

The square root of x is obtained by using the squaring circuit as feedback to a high-gain amplifier.

7-7. GENERATION OF FUNCTIONS OF TWO VARIABLES

In the study of problems such as aerodynamic stability the aerodynamic coefficients are in general functions of more than one variable, for example: Mach number, angle of attack, angle of sideslip, etc. If a large simulation is to be conducted in which these effects are to be taken into consideration, then a method for generation of functions of two or more variables is required.

Perhaps the most elementary method for generation of a function of two variables consists in decomposing a given function of several variables into combinations of functions of single variables. This process is illustrated by a practical example which represents the rolling moment due to sideslip for a VTOL-type airplane. During the transition from horizontal to vertical flight the rolling moment $C_{l\beta}$ due to sideslip can be represented by Eq. (7-13).

$$C_{l\beta} = C_0(\alpha) + C_1(\alpha)\beta + C_2(\alpha)\beta^2 \tag{7-13}$$

Equation (7-13) has the form of a power series in β. The coefficients $C_0(\alpha)$, $C_1(\alpha)$, and $C_2(\alpha)$ are arbitrary functions of the angle of attack α of the aircraft. In arriving at Eq. (7-13) it was found that for any given angle of attack C_{l_β} could be represented very closely by three terms of a power series. The coefficients of the approximating power series varied arbitrarily with the angle of attack. A large amount of preliminary numerical work was required. In this case, however, a large-scale digital computer was employed for the curve-fitting process. As a result arriving at an approximating equation such as Eq. (7-13) becomes largely a matter of routine. More complicated cases may require carrying the power series out to include higher terms. The programming of Eq. (7-13) is carried out in a straightforward manner by applying function generators of a single variable and suitable multiplication devices. If the power-series representation is carried out to include higher terms, difficulty will be experienced with scaling and accuracy. While it is theoretically possible to extend this scheme to the representation of functions of three variables or more, it is obvious that the numerical difficulties mount to staggering proportions. Programming the resulting approximation equation may require a prohibitively large amount of computing equipment.

The diode function generator can be made to generate functions of two variables. When used to generate functions of a single variable, the diode network acts as a variable impedance for different ranges of the input voltage e_i, thus making the output a nonlinear function $e_o = f(e_i)$, which is composed of segments of straight lines. The break points between adjoining segments are determined by fixed-bias voltages applied to the diodes in the network. To generate functions of two variables, the constant-bias voltages are replaced by variable voltages proportional to y or a function $g(y)$ so that the network represents a variable impedance for different ranges of the input variables e_i and y. Thus the original output curve $e_o = f(e_i)$, obtained with fixed bias, is modified in a continuous manner into a family of curves with parameter y. By an appropriate synthesis of the diode network these curves can be shaped to approximate a desired function $f(e_i,y)$. The method can be further extended to functions of more than two variables.

Consider the application of the circuit of Fig. 7-11 to the generation of functions of two variables, $f(e_i,y)$, for the case where the fixed-bias voltage y is changed successively to y_1, y_2, y_3, etc., where $y_1 > y_2 > y_3$. Now, since $y_2 < y_1$, the point where conduction of the diode begins will occur at a smaller value of e_i, that is, $e_{By_2} > e_{By_1}$. Therefore the break points B_i are displaced in the negative e_i direction. This causes the original curve to be deformed progressively into the curves shown in Fig. 7-20, which have y_1, y_2, y_3, etc., as parameter values. Thus a func-

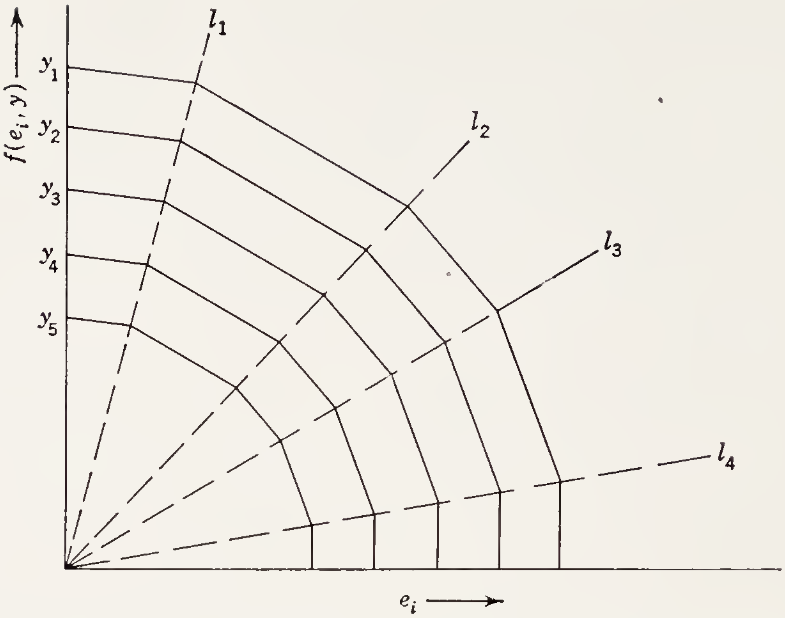

FIG. 7-20. Simple function of two variables produced by diode circuit of Fig. 7-11.

tion $f(e_i,y)$ is obtained. In this process the segments C_0, C_1, C_2, . . . , C_n of the original curve are translated into corresponding segments of the new curves for $y = y_2, y_3, y_4, \ldots, y_n$ since the slopes are not affected by the change in y. In the simple case considered, equal increments in y cause uniform displacements of the segment C_0 and the break-point voltages e_{Bi} so that each break point B_i moves on a straight line l_i. The loci of break-point shift have a common intersection at the origin of the e_i, $f(e_i,y)$ plane. This example shows that e_i and y voltages affect the diode circuit and function-shaping process in an entirely symmetrical manner. Either of these voltages may be considered as the bias voltage with respect to the other voltage as input.

The example considered so far requires a uniform shift of the segments for equal increments in y. Suppose now that we consider the family of curves generated by feeding a nonlinear function $g(y)$ instead of y to the diodes and the summing amplifier. The break-point voltages thus change nonuniformly in the same manner as the position of the C_0 segment. In this case, however, the break-point voltages are still along straight-line loci whose common intersection is at the origin. The distance between curves of the family varies nonuniformly according to the function $g(y)$. In a more general case the shift of the initial slope C_0 and break-point shift are determined by independent nonlinear functions $h(y)$ and $g(y)$, respectively. In general, this case will result in the loci of break points intersecting at some place in the e_i, $f(e_i,y)$ plane other than the origin. Otherwise this case resembles the one previously discussed.

In the most general case $f(e_i,y)$ is of completely arbitrary shape, and the slopes from one family y_i to y_{i+1} bear no relationship to each other. To generate such functions, independent y input functions $g_1(y)$, $g_2(y)$, . . . , etc., are required for different diode channels. The total amount of function-generating equipment then may become quite large.

A method for the generation of functions of two variables has recently been developed using a servo-driven plotting board with a sheet of semiconducting material substituted for the plotting surface and with an insulated pickup probe substituted for the pen. The probe is positioned by the input voltages to the plotting-table axes and is held in contact with the semiconducting surface by a spring. Since the plotting board is an electromechanical device, this method of function generation is restricted to those problems whose frequency-response requirements are less than 2 to 3 cps.

The function of two variables is represented by plotting lines of constant value on the semiconducting surface with conducting silver paint. Each line is then held at a voltage corresponding to the value of the function for that line. The surface electrically interpolates these voltages between lines, and thus the voltage on the probe corresponds to the desired function of the x and y inputs to the plotting board. The accuracy of interpolation between conducting lines depends upon the nature of the function to be generated. Figure 7-21 illustrates the setup of the semiconducting plate for a function involved in the performance of a

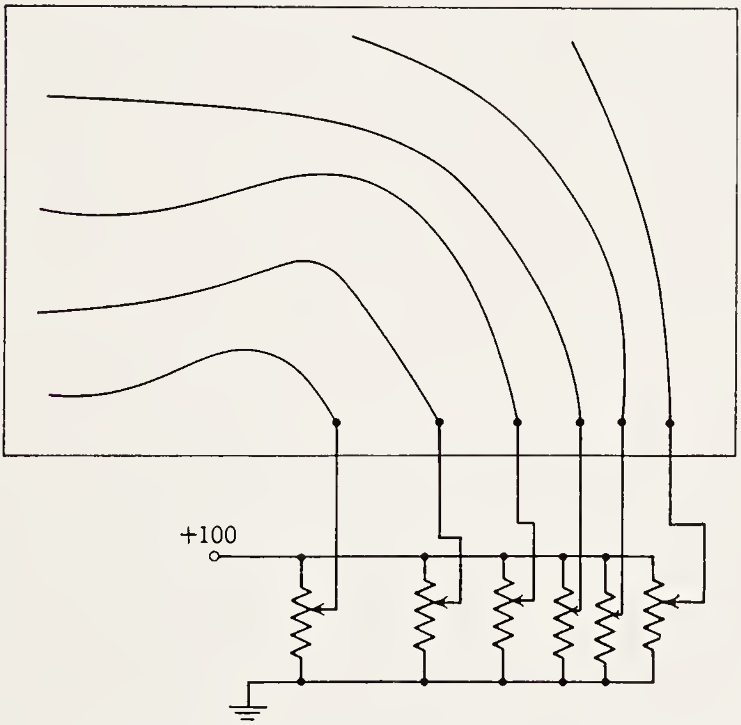

FIG. 7-21. Setup of semiconduction plate for generation of a function of two variables, including voltage sources.

turbojet engine. The voltage sources for supplying the constant potentials are also shown in Fig. 7-21. In the form shown, difficulty would be experienced in calibrating the voltage sources owing to interaction between the settings of the source-voltage potentiometers. This interaction is caused by current flow from one line to the next through the semiconducting material. The problem can be overcome in practice if, during the calibration process of line i, lines $i - 1$ and $i + 1$ are driven temporarily by amplifiers whose output voltages are set to the values desired for lines $i - 1$ and $i + 1$.

Several materials can be used for the semiconducting sheet. Among these are Teledeltos paper, a conducting paint mixture on a glass surface, and an insulating plastic bonded to a hard resistance film. The latter material has recently been made available commercially by Electronics Associates, Inc.

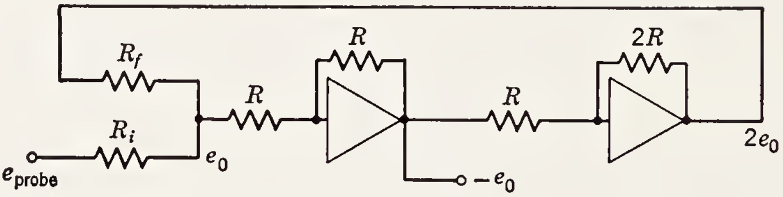

FIG. 7-22. Unloading, or isolation, amplifier.

Because of the high resistivity of the conducting surface, loading effects due to probe current are significant. To eliminate this effect, an unloading amplifier circuit (Fig. 7-22) is employed. The performance of this circuit can be analyzed by means of the nodal equation for the input to the first amplifier. Let i_p be the probe current; then, assuming the grid of the amplifier to be at zero potential, we have that

$$i_p = \frac{e_o}{R} - \frac{e_o}{R_f} = e_o \frac{R_f - R}{RR_f} \tag{7-14}$$

and the impedance seen by the probe voltage e_{probe} is

$$Z = \frac{e_{\text{probe}}}{i_p} = \frac{e_o}{i_p} + R_i = \frac{RR_f}{R_f - R} + R_i \tag{7-15}$$

Hence, if $R = R_f$, the input impedance is infinite.

Using a single probe, a slight error in output voltage is introduced, due to the width of line and probe. As the probe passes over a constant-voltage line, the output voltage remains constant for the interval in which the probe touches the line. To avoid this undesirable flat spot in output voltage, a method of averaging is used which employs four probes arranged at the corners of a square; the average of the four voltages, commonly connected to the unloading circuit through like resistances of several megohms, is used to approximate the value at the center of the square. As only a maximum of two probes can touch an equipotential line at any time, the effect of line and probe width is minimized and a smooth output curve is obtained.

To examine the nature of the interpolation between lines of constant potential, an analysis must be made using Laplace's equation. For the special case of two parallel constant-voltage lines, interpolation between lines is linear. For functions whose lines of constant value on the x, y plane are sharply curved, the electrical interpolation of the semiconducting surface of the voltages between the lines can be a major source of error. Depending on the nature of the curve to be approximated, the electrical interpolation can be curved in a direction opposite to that in which the function to be generated is curved. This point is illustrated in Ref. 3, where the interpolation is worked out for a pair of concentric circles and the result applied to simulate a hemisphere. The hemisphere has decreasing curvature and the interpolation increasing curvature; hence the resulting approximation to the hemisphere is as illustrated in Fig. 7-23. In general, the variation in potential on a uniform semiconducting surface, in a region containing no sources or sinks, is specified by Laplace's equation $\nabla^2 V = 0$. For two dimensions x and y this may be written as

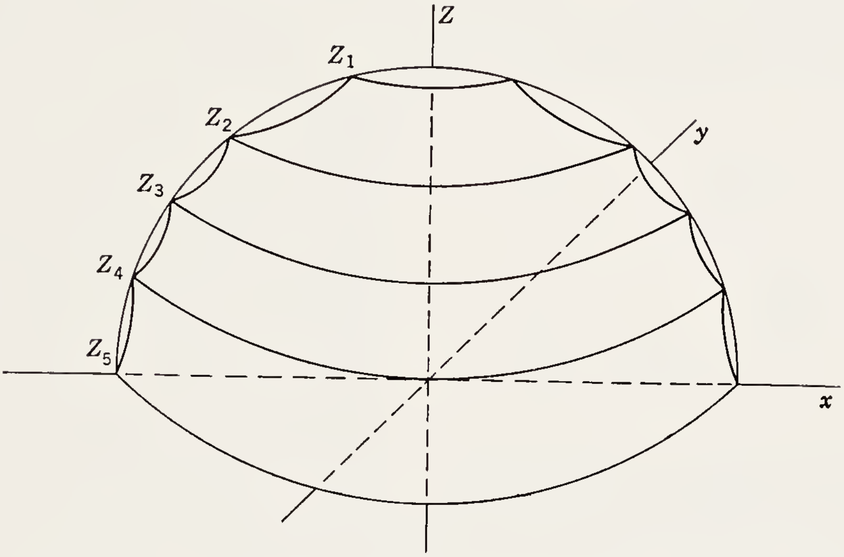

FIG. 7-23. Simulation of a hemisphere in which the interpolation between lines of constant potential has opposite curvature to the function generated.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{7-16}$$

The term $\partial^2 V/\partial x^2$ represents the rate of change of slope of the function

V in the x direction, and the term $\partial^2 V/\partial y^2$ represents the rate of change of slope of the function V in the y direction. The equation indicates that, at any point, the rate of change of slope in the x direction must be the negative of that in the y direction. The more nearly the function F, to be represented, fulfills these conditions, the more accurately it may be generated by this type of function generator.

7-8. IMPROVING THE ACCURACY OF DIODE FUNCTION GENERATORS

The fundamental principle of operation of diode function generators is based on using the diode as a voltage-sensitive switch. Thus the curve generated is a straight-line approximation to the arbitrary function desired. In practice, the diode does not represent a perfect switch. Passage from the nonconductive to the conductive state occurs gradually over a small region. This causes the diode to round off the sharp break associated with the straight-line approximation. This effect is illustrated in Fig. 7-24. The fit obtained to the desired arbitrary function is thus somewhat better than the straight-line approximation.

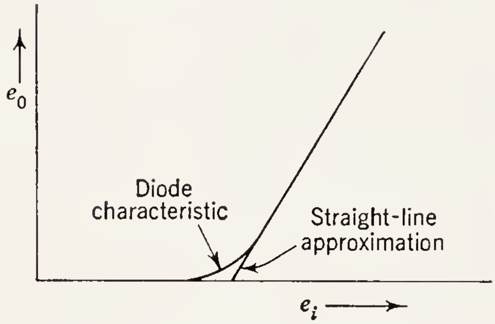

FIG. 7-24. Comparison of diode characteristics with straight-line approximation.

A further smoothing effect may be obtained by adding to the input a signal from a smoothing oscillator. Suppose that a high-frequency sine wave is superimposed on the d-c level of the input signal (see Fig. 7-25). Instead of cutting into the switching region of the diode sharply,

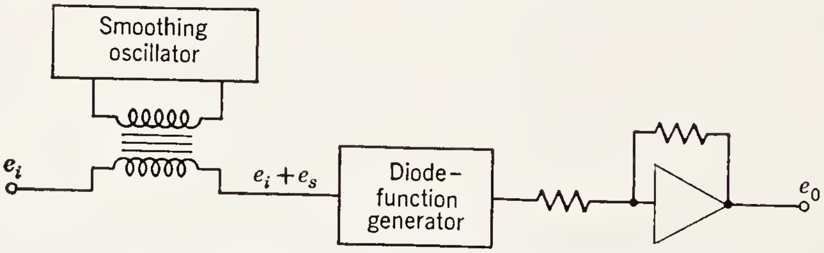

FIG. 7-25. Addition of smoothing oscillation to input signal.

as in the case of straight-line segment generation, there is added to the output a voltage due to the diode commencing conduction periodically. As e_i increases nearer the break voltage, the mean value of the voltage added by e_s increases to a maximum, when e_i equals the break voltage e_B. As e_i continues to increase past this point, the mean value of the voltage contributed by e_s decreases, until it is zero when $e_i = e_B + e_s$. At this point the oscillatory component through any one given diode is symmetrical, and the mean value, or d-c component, is therefore zero. Figure 7-26 shows the effect of this additional smoothing. Instead of having the straight line and a definite break point, the smooth curve departs from the straight line at an input voltage which depends upon the amplitude of the smoothing-oscillator output. As indicated in Fig. 7-26, the output transformer of the oscillator is connected in series with the input voltage e_i.

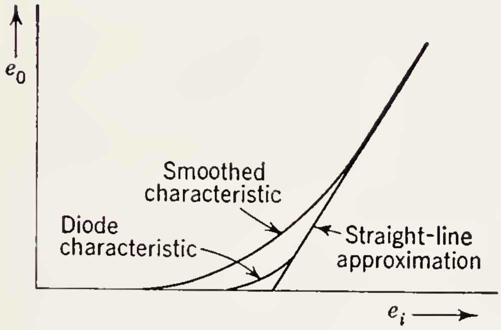

FIG. 7-26. Effect of smoothing on the voltage near a break point.

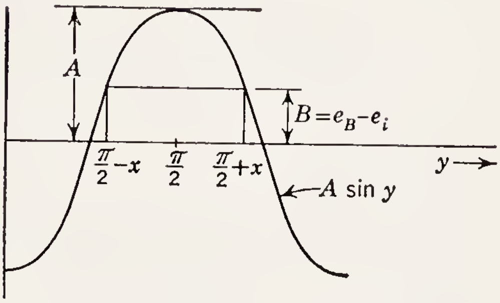

FIG. 7-27. Definition of conduction angle.

The magnitude of the voltage contributed by the smoothing oscillator can be calculated by considering the average value, or d-c components, in e_o due to e_s. We assume ideal diode characteristics. The average value of a periodic function is defined as the area between the axes of abscissa and the function, for one period, divided by the length of the period. If a sinusoidal function of magnitude A and frequency ω is added to the input voltage e_i, then the input to the diode is $e_i' = e_i + A \sin \omega t$. To calculate the average value of the output voltage due to the a-c component of e_i', we note that, if $e_i' > e_B$, the bias voltage, at any time during the oscillation of e_s, the diode will conduct. During the conduction period the emf equation for the oscillatory component is

$$e_{o_{\text{a-c}}} = \frac{-R_f}{R_i} A \sin \omega t \tag{7-17}$$

The average value is computed by considering Fig. 7-27. From this figure, if we let $\omega t = y$, then

$$\begin{aligned} e_{o_{\text{av}}} &= -\frac{1}{2\pi}\frac{R_f}{R_i}\int_{\pi/2-x}^{\pi/2+x} (A \sin y - |B|)\, dy \\ &= -\frac{R_f}{R_i}\frac{1}{\pi}(A \sin x - |B|x) \end{aligned} \tag{7-18}$$

where the conduction angle $2x$ of the diode is defined as that fraction of the cycle over which the diode is conducting. R_f and R_i are the feedback and input resistances of the amplifier, respectively.

From Fig. 7-27 it is apparent that

$$B = e_B - e_i = A \sin\left(\frac{\pi}{2} + x\right) \tag{7-19}$$

and

$$x = \arcsin \sqrt{\frac{A^2 - B^2}{A}} \tag{7-20}$$

Substituting Eq. (7-20) in Eq. (7-18), there is obtained

$$e_{o_{\text{av}}} = -\frac{R_f}{R_i}\frac{1}{\pi}\left(\sqrt{A^2 - B^2} - |B| \arcsin \frac{\sqrt{A^2 - B^2}}{A}\right) \tag{7-21}$$

This may be normalized with respect to the amplitude of the input sinusoid, to give

$$Z = \frac{\pi e_{o_{\text{av}}}}{A} = -\frac{R_f}{R}\left[\sqrt{1 - \left(\frac{B}{A}\right)^2} - \frac{|B|}{A} \arcsin \sqrt{1 - \left(\frac{B}{A}\right)^2}\right] \tag{7-22}$$

A most interesting case occurs when the smoothing oscillation is made to have the form of a triangular wave. It can be shown that for this case the average of the voltage contributed by the triangular wave is

$$e_{o_{\text{av}}} = -\frac{R_f}{4R_i} A \left(1 - \frac{|B|}{A}\right)^2 \tag{7-23}$$

where A is the half amplitude of the triangular wave and $B = e_B - e_i$. This may be normalized with respect to the amplitude of the triangular wave, to obtain

$$Z = \frac{4e_{o_{\text{av}}}}{A} = -\frac{R_f}{R_i}\left(1 - \frac{|B|}{A}\right)^2 \tag{7-24}$$

In the case of both sine-wave and triangular-wave smoothing the smoothing oscillation makes a contribution for the region where $-A < B < A$ and is a maximum when $B = 0$. This occurs when the input $e_i = e_B$. Figure 7-28 is a plot of the normalized Eqs. (7-22) and (7-24). Also shown in this figure is the normalized contribution from a square-wave oscillator. From Fig. 7-28 it is apparent that if a diode function generator is used, in which square-wave smoothing is applied, any two-line-segment portion of the approximating curve is replaced by a three-line-segment fit over the same portion of the curve. In the case of sine-wave and triangular-wave smoothing the interpolation between break points follows a smooth curve.

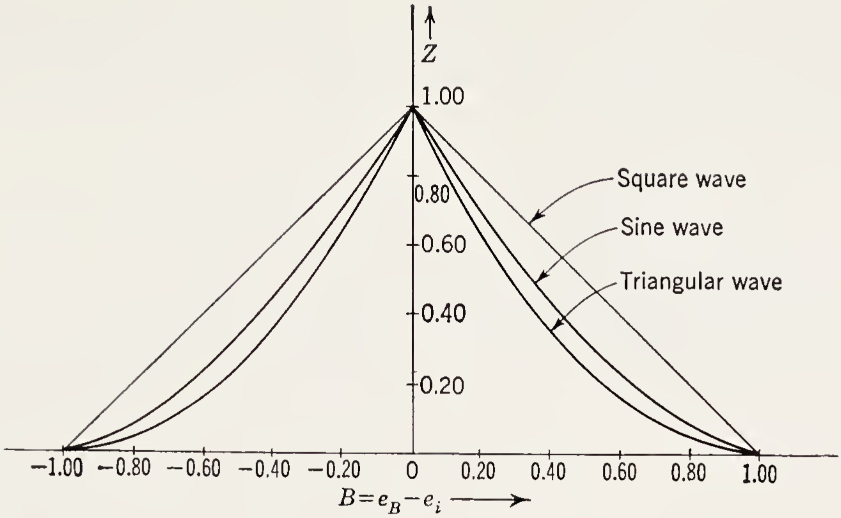

FIG. 7-28. Normalized contribution to output voltage from smoothing oscillator.

Suppose that we consider the circuit of Fig. 7-29. In this figure a triangular-smoothing oscillation of double amplitude of 100 volts is superimposed on the input of a pair of oppositely biased diodes. The diodes are set to conduct when the bias $|e_B| = 50$ volts. The gain of the output amplifier is set so that the ramp function obtained, after either of the diodes fires, has a slope of 2. Now, since the half amplitude of the triangular oscillation is set to 50 volts, then, according to Eq. (7-23), the net output of the circuit will be exactly $-e_i^2/100$ for $-100 \leq e_i \leq 100$.

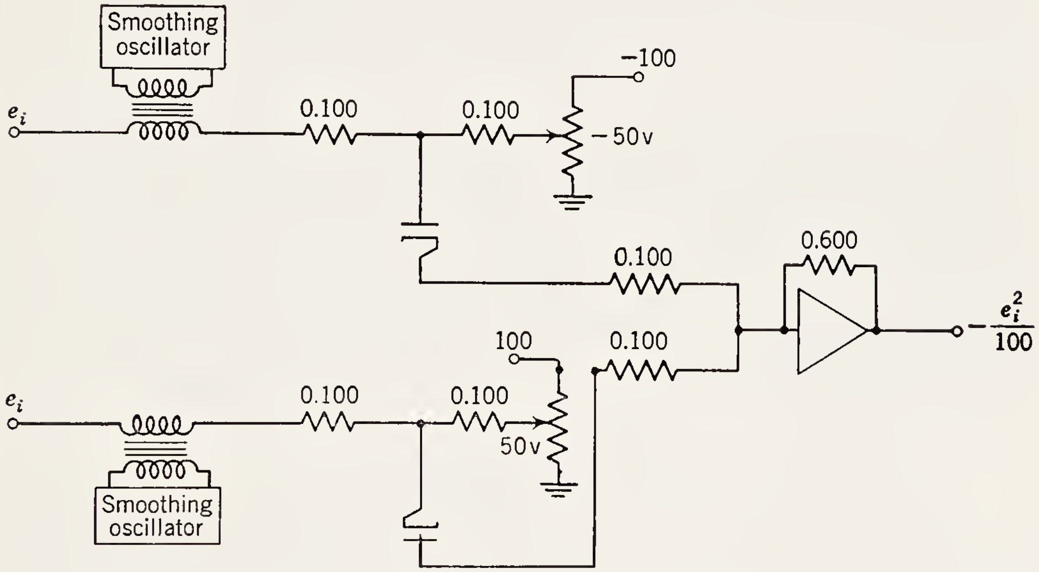

FIG. 7-29. Use of triangular-wave smoothing oscillator to generate $e_i^2/100$ exactly.

Since in the circuit of Fig. 7-29 the square function is generated exactly, an excellent theoretical basis for the construction of a high-accuracy quarter-square multiplier has been established. The complete multiplier circuit is shown as Fig. 7-30.

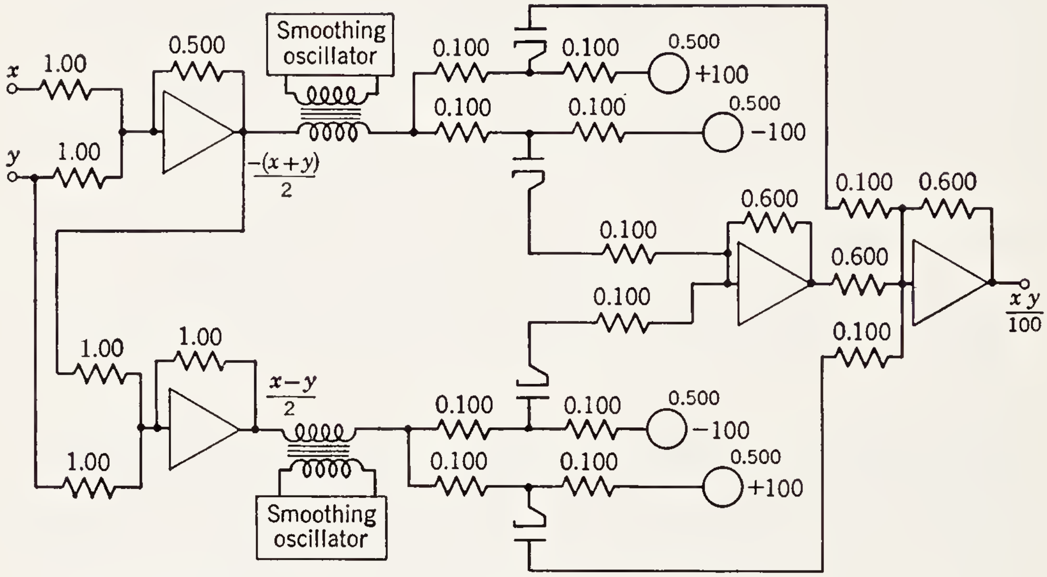

FIG. 7-30. Four-amplifier quarter-square multiplier using smoothing oscillator.

PROBLEMS

7-1. Plot the curve $y = 20 \sin x + 10 \sin 2x$ for $-\pi/2 \leq x \leq \pi/2$.

a. Determine a scale factor to utilize the major portion of the ±100-volt range.

b. Make a straight-line fit to the curve with 16 straight-line segments for use on a 17-tap tapped potentiometer.

c. Make a 10-segment approximation to the curve to be used with the circuit of Fig. 7-11.

7-2. Using a circuit similar to that of Fig. 7-19, devise a circuit for computing the square root of a positive voltage.

7-3. Using Fig. 7-14 and given α and β, devise a circuit for generating

$$\psi = \arcsin \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}$$

7-4. Show that the average value contributed by a triangular smoothing oscillation is given by

$$e_{o_{av}} = -\frac{R_f}{4R_i} A \left(1 - \frac{|B|}{A}\right)^2$$

where the symbols are the same as those used in the text.

7-5. The isolation amplifier of Fig. 7-22 can be generalized to the circuit shown in

Fig. P7-5. Determine the value of the resistor R_F such that the impedance seen by the probe is infinite.

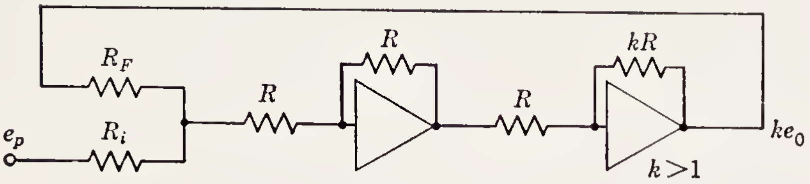

Fig. P7-5.

7-6. Let $U = -[(x + y)/2]$ and $V = -[(x - y)/2]$, and, assuming function generators available which generate

$$g = -\frac{(U - 100)^2}{200}$$

$$h = \frac{(V + 100)^2}{200}$$

devise a three-amplifier circuit capable of performing four-quadrant multiplication.

7-7. Using the technique of Fig. 7-29, devise a function generator for computing g and h of Prob. 7-6.

7-8. Determine the formula for interpolation between two concentric circles for the two-variable function generator of Fig. 7-21.

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CHAPTER 8

General Theory of Operation of Computer Components

8-1. INTRODUCTION

Up to this point a great many assumptions have been made as to the characteristics of computer components and with regard to the techniques of computer control. In Chap. 1 the concept of the d-c operational amplifier was introduced without giving a description of its operation. Certain properties were postulated for the amplifier, and, based on these properties, the numerous operations presented in the subsequent chapters were developed. We shall consider in this chapter the properties of the d-c operational amplifier in detail and shall determine in what manner this device deviates from the ideal. We shall examine the types of relay systems used for computer control and shall investigate means for (1) starting and stopping the computer at will; (2) introducing initial conditions; (3) stopping the problem at any point and holding the solutions static; and (4) introducing or disconnecting reference voltages at will. In addition to the above standard modes of operation, we shall consider automatic techniques for (1) the checking of accuracy of problem setup; (2) time-scale check of problem setup, i.e., the determination of problem and computer response compatibility; and (3) the setup of computer problems so that multishift operation becomes a practical reality.

8-2. THE D-C OPERATIONAL AMPLIFIER

An amplifier is defined as a device which, when an input voltage e_o/μ is applied, yields for its output a voltage e_o. The factor μ, the amplification factor, or gain, is the ratio of the output voltage to the input voltage. In the idealized amplifier the gain μ is constant for all frequencies, and the amplifier may be assigned the transfer function $e_o/e_i = \mu$. In the d-c operational amplifier a sign reversal is built in so that, for low enough frequencies, it has the transfer function $e_o/e_i = -\mu$.

In analyzing the circuits presented in the previous chapters of this book it has been assumed that the gain of the d-c operational amplifier was infinite. This assumption resulted in simplification of the relations governing the performance of the circuits. We wish now to examine the implications of this assumption. Consider the circuit of Fig. 8-1 for a summing amplifier. Assuming infinite amplifier gain μ, the voltage changes required at the input-grid terminal will be zero whatever the changes in e_o. During normal computer operation the input-grid voltage remains constant and is at ground potential. Suppose that an input voltage e_i is applied to the input resistor R_i. As a result of the postulated infinite amplifier gain, the response in e_o will be infinite, and an infinite current will be drawn from the source via the resistor R_f. If an infinite current flows in response to a finite voltage change, the effective resistance at the input-grid terminal must be zero. Thus we have established that under the assumption of infinite gain the input-grid terminal must remain at ground potential. For this reason the input grid of a d-c operational amplifier behaves as though it were actually connected to ground. This concept has been applied throughout this book and is extremely useful, since the input-grid potential always has a known value. In simple terms the action of the summing amplifier can be looked upon in the following manner: The current flowing in the input resistance R_i is $(e_i - 0)/R_i$, and the current flowing through the feedback resistor is $(e_o - 0)/R_f$. Now the amplifier is assumed to be working in its linear region, and there is no current flowing to the input grid. In this case all the current flowing in the input resistor toward the input grid can be assumed to flow on and up through the feedback resistors. The action of the summing circuit is thus to adjust the output voltage until the input-grid potential is zero.

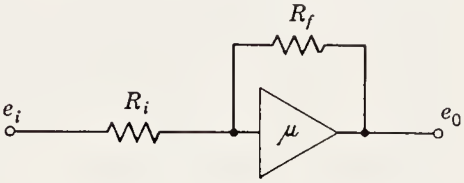

FIG. 8-1. Summing amplifier circuit.

What is the effect on computing accuracy of the finite gain μ? In practice the gain μ is not truly infinite but has a value which ranges from 5×10^7 to 5×10^8. A gain of this order of magnitude is sufficiently high to ensure that the error introduced due to the finite gain is negligible compared with the errors introduced by the best passive-element computing components available. Thus, if a gain of 100 were required by an amplifier whose open-loop gain is 5×10^7, the error due to the finite gain would be, by Eq. (1-4),

$$101 \frac{e_o}{\mu} = \frac{101}{5 \times 10^7} e_o \doteq \frac{e_o}{5 \times 10^5}$$

The maximum value for $e_o = 100$, so that the maximum voltage would be $1/5{,}000 = 2 \times 10^{-4}$ volt. Gain has thus lost its meaning as a criterion for evaluating amplifier performance. For, no matter how much gain the amplifier in an integrator or summer may have, the over-all performance cannot be better than the limits imposed by the passive elements. The best passive elements available today are such that raising the amplifier gain above about 10 million will make no further improvement in accuracy.

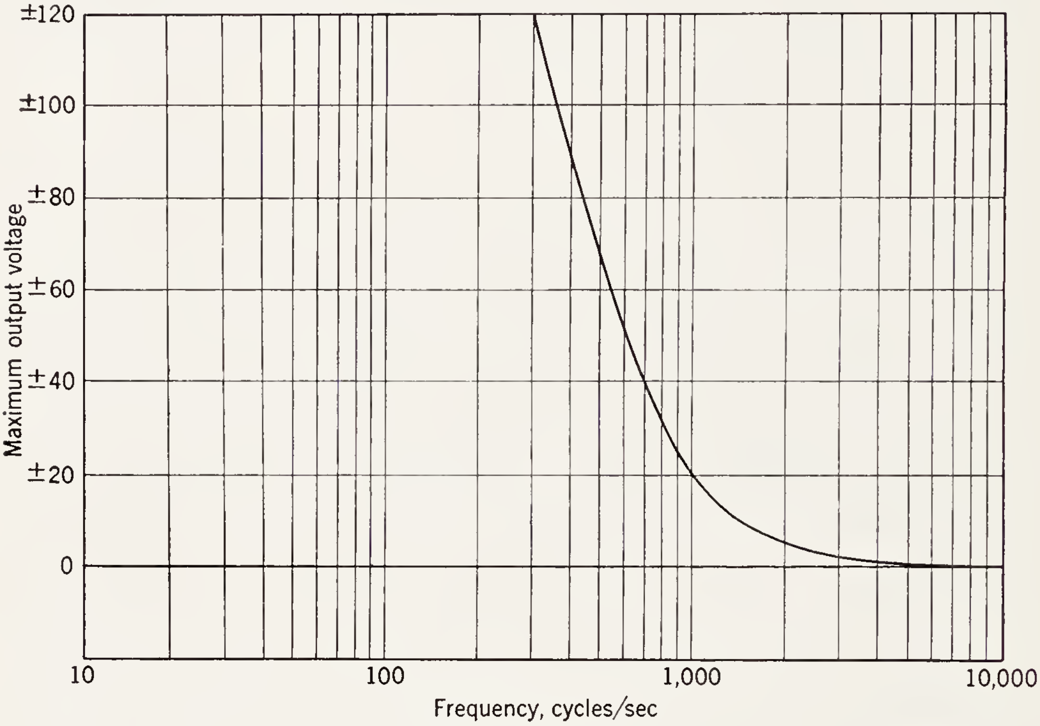

FIG. 8-2. Typical amplifier overload characteristics, PACE amplifier 16-6H.

Ideally the d-c operational amplifier should give an output voltage which is an exact amplified image of the input voltage, and, in order that a high degree of feedback can be applied by simple means, the amplifier should give a reversal of sign between input and output. Practical amplifiers fail to give this ideal performance for a number of reasons, chief among which are the effects of nonlinearities and of unwanted drift voltages combined with the desired voltages. The nonlinearity appears as curvature of the graph relating input and output voltages; i.e., the gain μ does not remain constant when e_i varies. This curvature tends to increase rapidly as the output voltage approaches the "overload" value.*

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* Whenever the absolute value of the output voltage of an operational amplifier exceeds a certain permissible level, the amplifier will saturate and can then no longer function as a linear device. When in this condition, the amplifier is said to be overloaded.

For most modern computing amplifiers this curvature does not occur within the normal ±100-volt operating range at normal computing frequencies. Most amplifiers will allow linear computations for voltages which exceed the operating range by 20 to 50 per cent. For high computing frequencies the situation is somewhat different. The overload characteristics fall off rapidly with frequency. Figure 8-2 shows typical overload characteristics of the Electronics Associates d-c amplifier, type 16-6H. As is evident from this figure, the allowable linear operating range of the computing amplifier decreases rapidly for frequencies over 350 cps. Fortunately in most problems the computing frequencies are usually less than 100 cps, and the fall-off of the overload characteristic at high frequencies can be neglected.

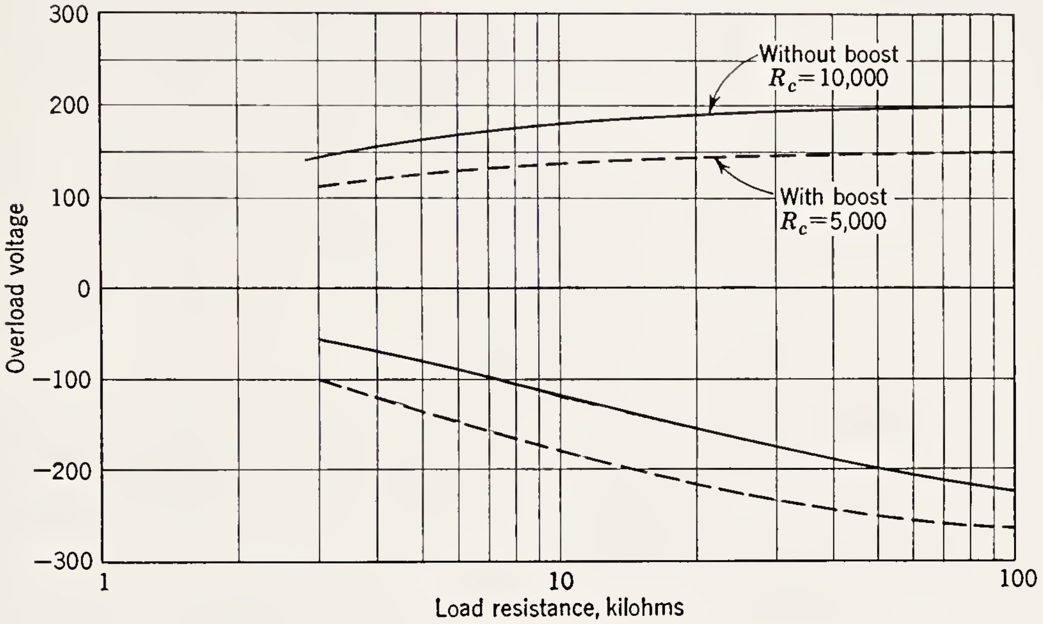

Fig. 8-3. Overload voltage characteristic for PACE amplifier 16-6H.

A more serious source of nonlinearity of the amplifier-gain characteristic comes about owing to the overload resulting from excessive output current. Figure 8-3 shows the amplifier-output overload-voltage characteristic for the 16-6H PACE amplifier. The solid curve gives the characteristic for the normal cathode resistor R_c = 10,000 ohms. The overload-voltage characteristic is determined by the maximum power dissipation allowable, with the amplifier output still a linear function of the input voltage. It will be noticed that this curve is not symmetrical with respect to the abscissa, being greater for positive than for negative voltages. If the cathode resistor is changed to a value, say 5,000 ohms (which is equivalent to boosting the amplifier), the second set of curves of Fig. 8-3 is obtained. It will be noticed that boosting the amplifier is equivalent to shifting the ordinate of the voltage-overload characteristic negatively.

When the input voltage of a d-c amplifier is kept constant, the output voltage changes with time from the original value to some other value. This phenomenon, called drift, is caused mainly by changes in plate and heater voltages, variations in resistor values, tube aging, and grid-current changes. In a multistage d-c amplifier drift occurring in an early stage causes a greater change in the output voltage than the same drift in a later stage because the early drift is amplified by the gain of all succeeding stages. If drift can be eliminated from the first stage, the output drift will be reduced in proportion to the gain of that stage. If the gain of this stage is considerable, the drift of the output will be reduced to a small fraction of its former value.

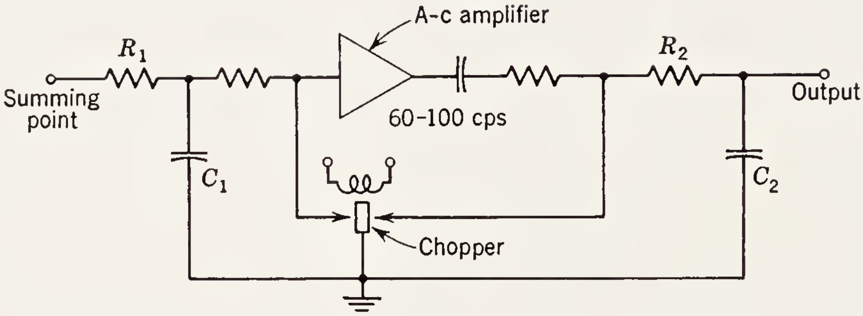

FIG. 8-4. Simplified diagram of the first stage of a typical d-c amplifier.

Two methods are in general use for the accomplishment of drift reduction in the first stage of amplification. These are individual-chopper-stabilized d-c amplifiers* and commutator-chopper-stabilized d-c amplifiers.† The principle by which the drift reduction is accomplished is the same in both systems. The major difference is that in one system each amplifier is supplied with an individual chopper, while in the other system the chopper is shared by several amplifiers by means of a commutator switch.

In the case of most of the individually chopper-stabilized amplifiers the first stage of amplification is an essentially drift-free amplifier with a gain of approximately 1,000. Figure 8-4 is a simplified diagram of the circuit. The chopper is simply an SPDT switch which switches back and forth at a frequency of from 60 to 100 cps. It alternately grounds the input and the output of the a-c amplifier. The resulting amplifier input is a square wave with one peak at ground potential and the other at a positive or negative voltage proportional to the voltage at the summing point. The output of this a-c amplifier is a greatly amplified square wave. The action of the chopper (which grounds the output during one-half of each cycle) places one peak of the amplified square wave at ground. This wave is then filtered by R_2 and C_2 to produce an average d-c voltage proportional to the d-c input. The resistance R_1 and capacitance C_1 form a filter which prevents 60-cps noise at the summing junction from reaching the a-c amplifier and producing a change in the d-c output level.

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* EASE, PACE, REAC.
† GEDA.

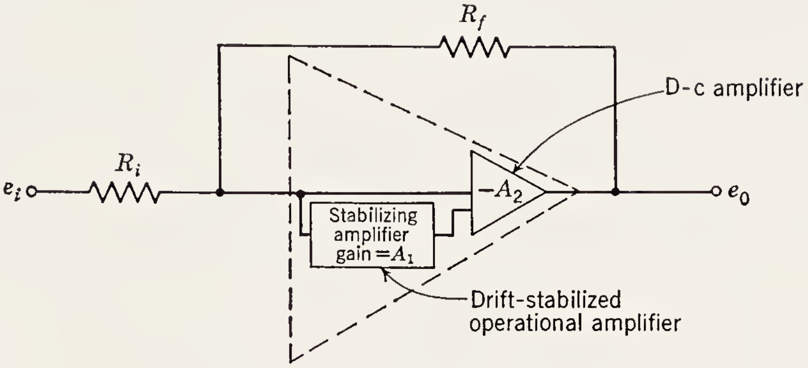
R_f
D-c amplifier
R_i
e_i
$-A_2$
e_0
Stabilizing amplifier gain $=A_1$
Drift-stabilized operational amplifier

FIG. 8-5. Connection of stabilizing amplifier into the operational amplifier circuit.

The output of the stabilizing amplifier is then applied to the input of the d-c amplifier. Figure 8-5 illustrates the arrangement. The input signal follows two paths: one directly to the d-c amplifier input, the other through the stabilizing amplifier to the d-c amplifier. The object of this arrangement is to combine the advantages of a drift-free first stage with the broad frequency response of a d-c amplifier. Figure 8-5 shows the connection of the stabilizing amplifier into the main operational amplifier. The stabilizing amplifier precedes the d-c amplifier as far as the d-c and low-frequency components of the signal are concerned. Since this channel is not affected by voltage changes in the plate-voltage and filament supplies, it acts essentially as an input stage having a relatively high gain A_1 and very low drift.

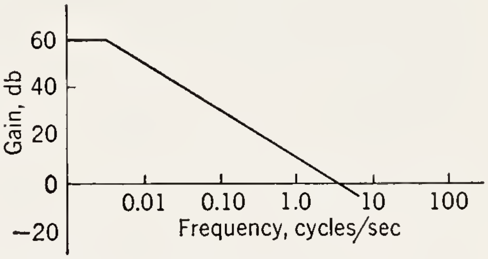

FIG. 8-6. Typical response characteristic of stabilizing amplifier.

The total gain of the operational amplifier is $\mu = A_2(1 + A_1)$, where A_2 is the gain of the d-c amplifier (60,000 to 100,000) and A_1 is the gain of the stabilizing amplifier. Owing to the output filtering network, the gain A_1 of the stabilizing amplifier falls off rapidly with frequency. Figure 8-6 shows a typical response characteristic for a stabilizing amplifier whose output filter consists of a simple RC network. At the drift frequencies (d-c to 0.005 cps) the stabilizing amplifier has a gain of about 1,000. Above 0.005 cps its gain decreases progressively until at higher frequencies this stage is effectively out of the circuit.

The function of a chopper-stabilized amplifier can be examined quantitatively. Since the effect of drift is most important in the case of the integrator, we shall derive the relationships for the case where an operational amplifier is used to perform integration. Figure 8-7 shows schematically a conventional chopper-stabilized integrator circuit. In this figure E_1 is the signal at the input to the summing junction of the amplifier, E_D is the amplifier unbalanced voltage due to drift, and i_g represents the input-grid current. The output voltage in terms of the inputs to the amplifier is

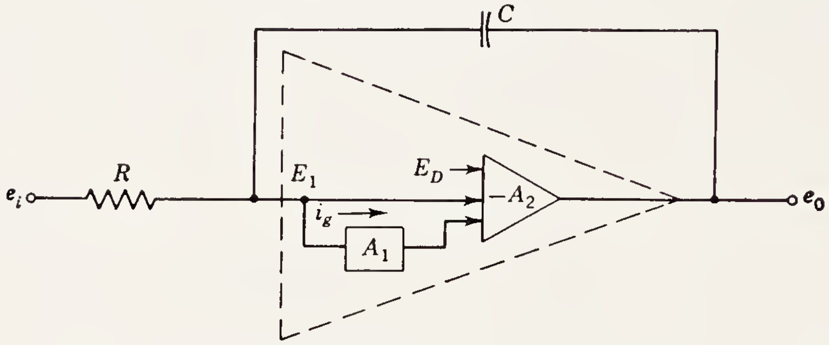

FIG. 8-7. Conventional chopper-stabilized integrator.

$$e_o = -A_2(E_1 + E_D + A_1E_1) \tag{8-1}$$

and, writing a nodal equation with the summing junction as the node, the voltage at the summing junction can be expressed as

$$E_1 = \frac{e_i + RCse_o - i_gR}{1 + RCs} \tag{8-2}$$

Combining Eqs. (8-1) and (8-2), we obtain

$$e_o = -A_2(1 + A_1)\frac{e_i + RCse_o - i_gR}{1 + RCs} - A_2E_D \tag{8-3}$$

Collecting terms and solving for e_o, we obtain

$$e_o = -\frac{A_2(1 + A_1)e_i}{1 + (1 + A_2 + A_1A_2)RCs} + \frac{A_2(1 + A_1)i_gR}{1 + (1 + A_2 + A_1A_2)RCs} - \frac{A_2E_D(1 + RCs)}{1 + (1 + A_2 + A_1A_2)RCs} \tag{8-4}$$

For A_2 large, this equation reduces to

$$e_o = -\frac{e_i}{RCs} + \frac{i_g}{Cs} - \frac{E_D(1 + RCs)}{(1 + A_1)RCs} \tag{8-5}$$

Equation (8-5) shows that the output voltage approximately equals the negative integral of the input, plus the integral of the input-grid current divided by C, plus a drift term. It is to be noticed that the drift term is divided by the gain of the stabilizing amplifier. Thus we have realized the reduction of the effect of drift by means of the stabilizing amplifier. The effect of the grid-current term can be reduced by two obvious methods. One method is to increase the value of C. However, this capacitor must be of the finest quality, and high-quality precision capacitors of large capacity are extremely expensive. A second method is to capacitively couple the input to the main d-c amplifier, allowing the chopper amplifier to maintain response at d-c and low frequencies, and thus blocking out the input-grid current, which is essentially d-c. This arrangement, however, suffers in that the input capacitor can be easily charged through the input tube, thus blocking the amplifier.

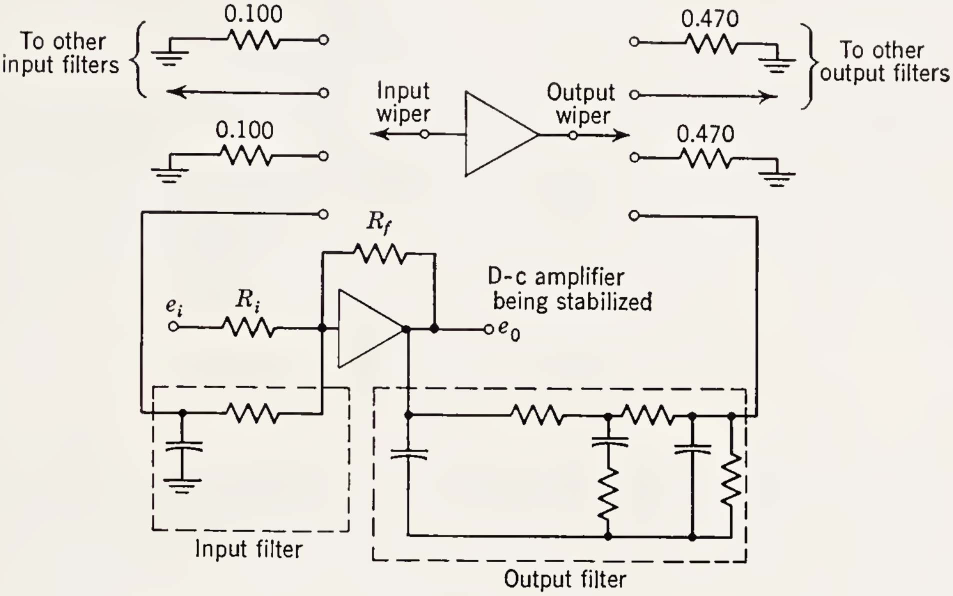

FIG. 8-8. Major parts of commutative chopper drift-stabilization system.

The commutator method for the reduction of the effect of drift in the first stage of amplification operates, in principle, the same as the individually chopper-stabilized amplifier. The major parts of the commutator-chopper-stabilized amplifier system are shown in Fig. 8-8. The summing-junction voltage of the d-c amplifier is fed to a commutator contact through an input filter. The input filter removes signals that are synchronous with the commutator sampling rate; these voltages otherwise would be synchronously rectified by the commutator and appear as offset in the output of the d-c amplifier.

The voltage applied to the commutator contacts is sampled by the commutator input wiper and amplified by the negative-gain stabilizer amplifier. This amplified voltage is then sent through the output wiper to the output filter of the same d-c amplifier. The output filter reduces the ripple resulting from the pulse voltages received from the commutator. The filtered voltage is applied to the balancing grid of the d-c amplifier, where it acts to restore the d-c balance of the amplifier.

In Sec. 1-2 the general dynamic equation [Eq. (1-13)] for the operational amplifier was derived. For the case of one input voltage this equation can be written as

$$E_o = -\frac{Z_f(s)}{Z_i(s)} \frac{\mu E_i}{1 + Z_f(s)/Z_i(s) + \mu} \tag{8-6}$$

In the application of this equation it has been assumed that the open-loop gain of the amplifier, μ, was independent of frequency. In actuality this is not the case, and the amplifier has dynamic characteristics and is a function of frequency. If $\mu(s)$ is the open-loop transfer function of the amplifier, then the closed-loop transfer function for a single input voltage is

$$\frac{E_o}{E_i} = -\frac{Z_f(s)}{Z(s)} \frac{\mu(s)}{1 + Z_f(s)/Z_i(s) + \mu(s)} \tag{8-7}$$

If the operational amplifier is to be stable, then the denominator of the closed-loop transfer function cannot have any zeros in the right half of the s plane. In the case of a summing amplifier where the input and feedback are resistors R_i and R_f, respectively, this means that, if the system is to be stable, $\mu(s)$ cannot take on the value $-(1 + R_f/R_i)$ for any s in the right half plane. Since $\mu(s)$ will normally not have any poles in the right-half s plane, this means that a Nyquist plot of the open-loop transfer function $\mu(s)$ must not encircle the critical point $[-(1 + R_f/R_i), 0]$. For a minimum-phase system $\mu(s)$, this in turn means that the Bode plot of the system must not have a slope as negative as -12 db/octave in the gain region round $1 + R_f/R_i$. To ensure that these conditions are not encountered for reasonable values of closed-loop gain R_f/R_i, shaping networks are employed. The dynamic characteristics which are of the most interest are those which involve using the amplifier as a summer. Figure 8-9 shows the closed-loop frequency response for REAC, PACE, GEDA, and EASE amplifiers for the condition of unity gain.

When the d-c amplifier is overloaded, the input summing grid is no longer at ground potential. This principle is utilized as the basis for construction of a computer overload-indicator system. This is usually accomplished by connecting a panel-mounted light across the output of the automatic balancing circuit. In operation, if, owing to an overload, the summing-junction voltage exceeds some predetermined value (10 to

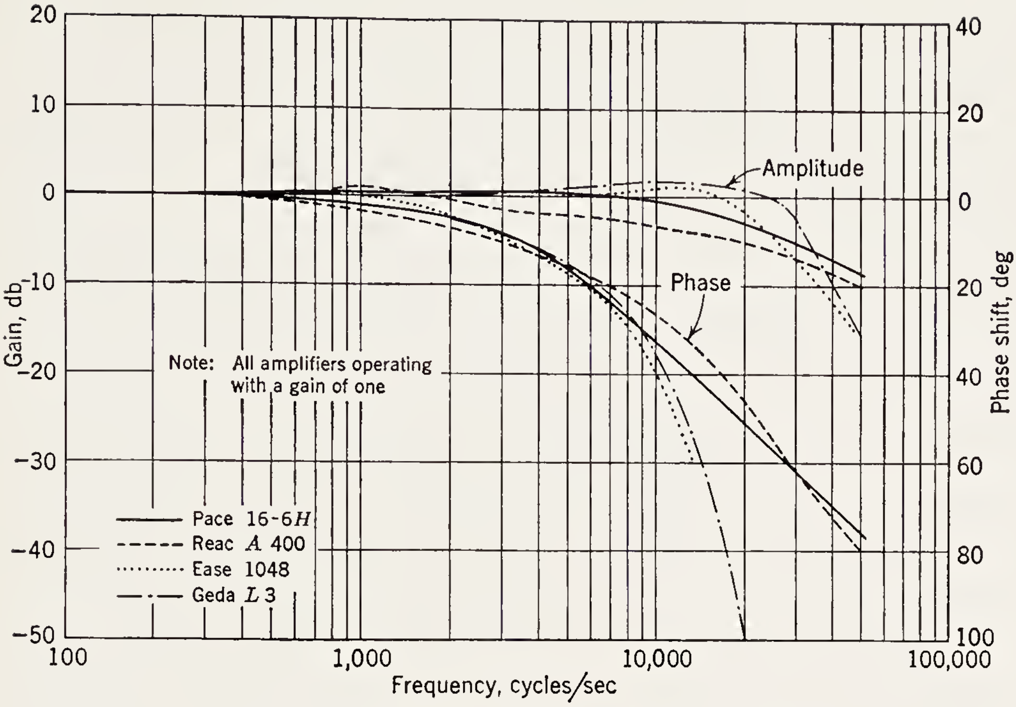

Fig. 8-9. Closed-loop frequency response of d-c amplifiers.

20 mv), the amplitude of the square wave out of the stabilizer amplifier becomes very large compared with its amplitude under normal conditions. This amplitude is of sufficient magnitude to light the overload indicator. If the summing-junction voltage is further increased, a point is reached where a circuit is energized to trip a relay which sounds an audible alarm. In some computers the audible-alarm relay can also be used to put the computer automatically in the "hold" condition.

8-3. COMPUTER CONTROL CIRCUITS

Before computation on an analog computer can begin, all the voltages representing variables must be set to correct values in accordance with the set of initial conditions associated with the problem. The basic function of starting and stopping a problem need involve only the integrators. The control of the computer is accomplished by means of a relay system which usually has four modes of operation. These are (1) the operate, or start, mode, whereby the inputs to the integrators are connected; (2) the stop, or hold, mode, which removes the inputs to the integrators and holds the problem variables at the value they had at the time of stopping; (3) the reset, or initial-condition, mode, which discharges the feedback capacitors of all the integrators and provides for introducing initial conditions on the integrators; and (4) the reference-off, or stand-by, mode, which removes the reference voltage from the problem board to permit changes in the problem programming. Some machines may provide computer control modes in addition to those already listed.

The four modes of operation of the computer controls will be illustrated by considering the relay system of the GEDA L3 computer shown as Fig. 8-10*a* to *d*. We shall consider the case for an integrating amplifier. The 1-megohm resistor shown connected to the inputs is a part of the input circuit "roll-off" networks and not a part of the external circuit. The 10-kilohm input and 1-megohm feedback resistors are connected to the amplifier by the stand-by relay when the programming switch is turned to stand-by. The summing points are grounded by other contacts on this relay so that the amplifier operates with zero input and a gain of 100 to prevent overloads due to stray voltages. This operation mode is also used for setting potentiometers. When the programming switch is thrown to the "initial-condition" position, the initial-condition relay closes and the stand-by relay opens. The closing of the initial-condition relay connects the 500-kilohm input and feedback resistors to the amplifier and applies the initial-condition voltage. When the programming switch is thrown to the "hold" position, the hold relay is energized and the initial-condition relay is de-energized. Energizing the hold relay disconnects the inputs to the integrators. In this position the inputs to the integrators are grounded, and the charge on the integrating capacitors remains constant. When the programming switch is thrown to the "operate" position, the hold relay is de-energized. De-energizing this relay connects the inputs to the amplifier.

A somewhat different arrangement of computer control circuits which allows repetitive computer operation* is illustrated by the system in the EASE model 1032 computer for the case of an integrating amplifier. The operating condition of the computer is established by the *A* and *B* relays serving all integrators. One side of the coils in these relays is connected through a special-integrate switch to points *A* and *B* in Fig. 8-11. As one or the other of these points is grounded or ungrounded, the corresponding relays are energized or de-energized. The operating-condition push buttons ground and unground points *A* and *B* in various combinations.

The compute condition is established by energizing both *A* and *B* relays. This connects the integrator input resistors to the summing junction. It also disconnects the initial-condition terminal from the

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* By repetitive computer operation we mean the operation of the computer on an accelerated time scale with provisions for resetting the computer variables to their given initial values periodically at repetition rates between 10 and 100 cps. Solutions of differential equations are thus repeated periodically and can be displayed on a cathode-ray oscilloscope whose linear sweep is synchronized with the initial-condition-setting circuits.

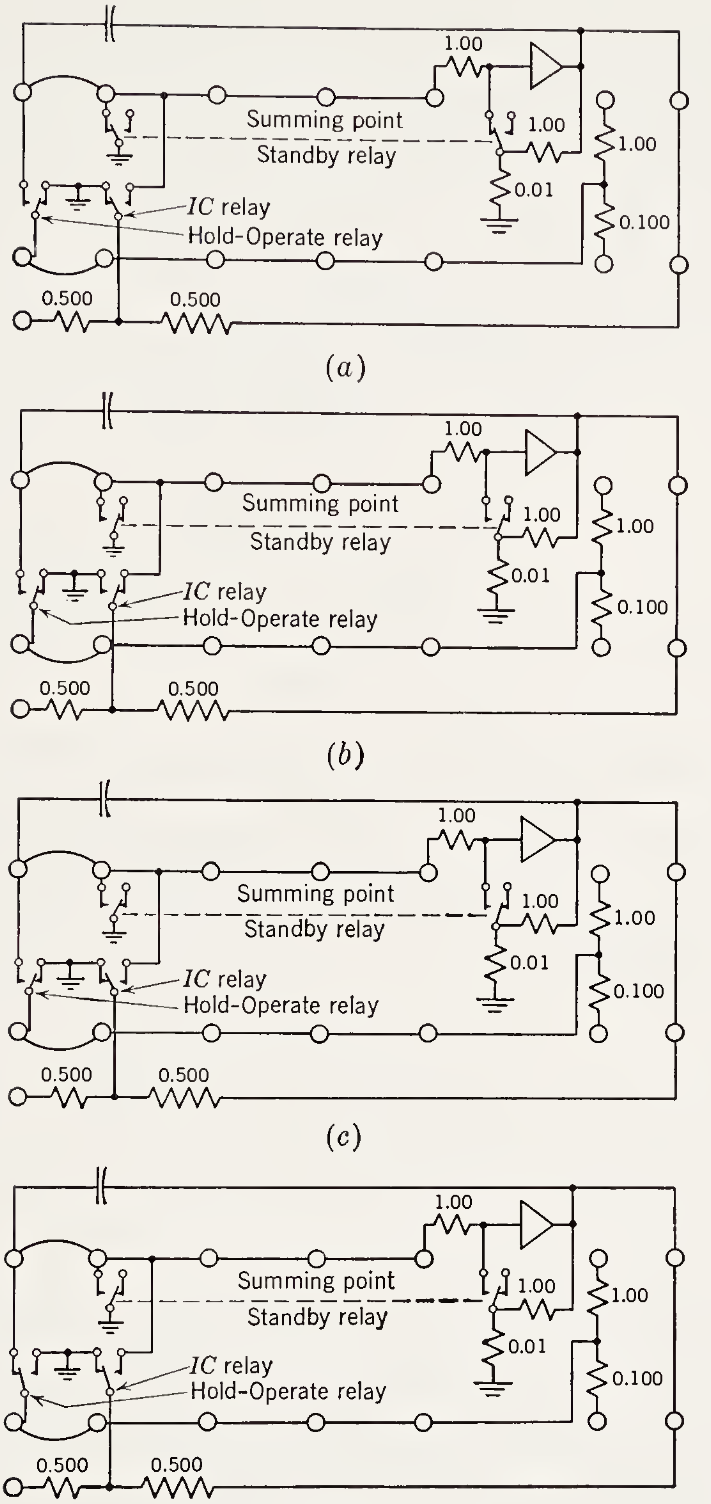

FIG. 8-10. Relay control circuit for GEDA computer. (*a*) Control switch in "standby" position. (*b*) Control switch in "initial-condition" position. (*c*) Control switch in "hold" position. (*d*) Control switch in "operate" position.

amplifier input grid, so that the initial-condition voltage is no longer applied across the feedback capacitor. In this condition the problem-solving circuit is complete, and the solution will be generated. Depressing the compute button grounds points A and B and energizes both relays.

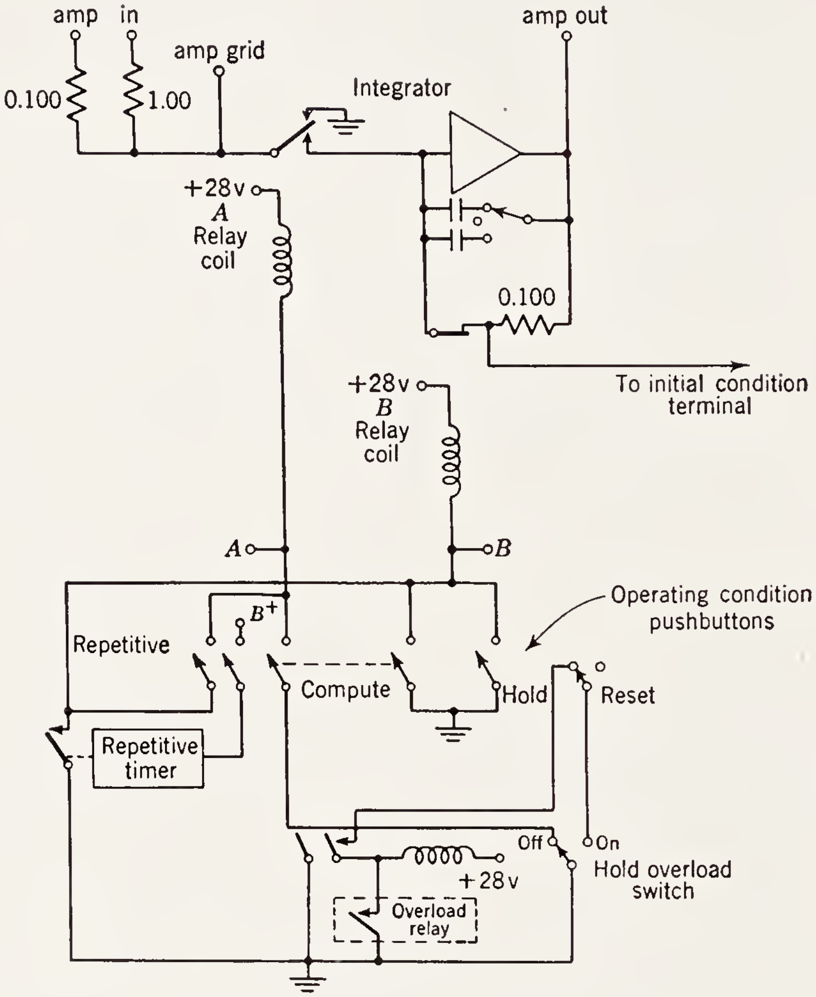

FIG. 8-11. Computer control circuit for EASE model 1032.

The hold condition is established by de-energizing the A relays while the B relays are left energized. This disconnects the integrator input resistors from the summing junction but leaves the feedback capacitors charged with the voltage built up during the compute period. All voltages throughout the problem-solving circuit are constant in this condition. Pressing the hold button grounds point B electrically and ungrounds point A by mechanically opening the compute-button switch.

The reset condition is established by de-energizing both A and B relays. This disconnects the integrator input resistors from the summing junction and grounds them. It also connects the initial-condition terminal to the amplifier grid so that any voltage patched across the amp-out and initial-condition terminals charges the feedback capacitor. If the initial-condition voltage is to be zero, no voltage source is connected across the amp-out and initial-condition terminals. In this case the 100-kilohm resistor simply discharges the feedback capacitor. Pressing the reset button ungrounds points *A* and *B* by mechanically opening all the other operating-condition push-button switches.

Depressing the repetitive button energizes the repetitive timer. The timer opens and closes a relay switch, thus repetitively grounding and ungrounding points *A* and *B* together. This switches the computer back and forth from "compute" (both points grounded) to "reset" (both points ungrounded).

8-4. AUTOMATIC CHECKING OF PROBLEM SETUP

One of the most serious criticisms made of analog computations has been that there was no simple and reliable method of checking the problem setup and the operation of the machine itself. Yet mistakes can be extremely costly in time and money. Patching a single wire into the wrong hole or a bad computer component, if not detected, may result in the loss of weeks of computer time. The computer carries an even greater responsibility in that, more often than not, the problems investigated are vital answers to large design problems. In this case investments ranging into the hundreds of thousands of dollars may depend on the accuracy of the computer results. A system of checking to ensure accuracy is thus desired.

The normal problem check consists of two phases: a static check of the problem and a dynamic check. In most computers the problem-check feature represents more positions on the computer control relay system. To understand the functioning of the problem-check system, we consider the computer setup for a typical differential equation. Such a setup represents a closed-loop system, and if it were not for the integrators, a static check would involve only algebraic manipulations. A simple comparison with the equations could then be made. Unfortunately, however, the integrator output on a static basis (that is, $t = 0$) is determined only by the initial conditions, and not at all by the signal inputs; so such a simple check is not possible with ordinary computer connections. If the computer control is thrown to the "problem-check static" position, two things are done. First, the output of each integrator is disconnected from its load, and a problem-check potentiometer is substituted in its place. The problem-check potentiometer is set to feed a voltage equal to the voltage of the integrator initial condition into the former integrator load. At the same time a 1-megohm feedback resistor is connected across the integrator, thus effectively converting it into a summing amplifier so that its output becomes the negative of the summation of the inputs. This is, in effect, the initial value of the derivative of the normal integrator output. As far as the rest of the circuit can tell, the loop is now complete and a position of static equilibrium is established. This equilibrium position represents the static solution of the equations for the particular initial conditions used. It is simply a matter of reading the voltage output of each amplifier and comparing it with values calculated from the problem equations to establish the correctness of the computer setup. This check is an extremely powerful one because it confirms the wiring of all computer elements and the correctness of potentiometer settings. Furthermore, it gives an indication of the static accuracy for all computer elements.

Now, only the correct operation of all integrators need be confirmed. For the "problem-check dynamic," or "integrator check," all integrators are converted back to true integrators and are returned to an initial condition of 0 volts. If now the integrators are allowed to integrate with the initial value of the derivative as input, then after a calibrated period of time say, 10 sec, the output will be 10 times the initial value of the derivative. In some integrator check systems a voltage is supplied through the same 1-megohm resistor previously used in the problem-check static condition as a feedback resistor. In the simplest version of this check feature, a relative check is made. The accuracy of all time constants is not determined absolutely, but relative agreement is found. The voltage is applied for a relatively long time interval through convenient relay circuits. Any error in time constant is detected by a deviation in output voltage from the remaining integrator amplifiers. A more complex version, in which a precise known voltage is applied for an exact time interval, determines the absolute value of the integrator time constant as well as a relative check.

8-5. AUTOMATIC TIME-SCALE CHECKING

Another problem which continually troubles users of analog computers is the question of suitable time scaling. Even the best of servos, electronic multipliers, function generators, etc., have frequency limitations, and if the time scaling is such that any one of the computer components is operating beyond its frequency range, an error will be introduced into the computer solution. To answer this question a computer control mode known as a "time-scale check" has been added to some computers.

The procedure is to run the problem once and in the normal fashion, recording the results on any convenient output device. The computer control switch is then thrown to the "time-scale check" position. In this position all time scales in the problem are changed in such a manner that the problem runs at exactly one-half the former speed (by doubling the value of all integrator capacitors). The problem is now rerun exactly as before and corresponding points on the two solutions compared. If the output device has two paper drive speeds which differ by a factor of 2, this comparison is especially simple. In this case the two solutions may then be laid one on top of the other and directly compared. It is apparent that, if the original time scaling is such that any computer component is contributing appreciable amplitude error or phase shift, the two solutions to the problem on the machine will differ.

8-6. AUTOMATIC-SETUP PROCEDURES

It is the fate of most computer centers to be overburdened with a variety of problems which must be solved concurrently. The magnitude of many of these problems is such as to strain the total computer capacity available. The time required for solution of any one of the larger problems may run into several weeks. Ordinarily setting up the potentiometers and initial conditions associated with the problem may run into several hours' work. If a multishift system of computer operation were attempted using ordinary hand-setup procedures, it would soon be found that a prohibitively large amount of time was being spent on setup and on tracing the errors induced by hand setup. One approach would be to set up a problem and leave it set up until completed, while employing different shift operators to work on the same problem on a multishift basis. This has not, in general, proved to be practical. The other approach is to install automatic-setup procedures.

Since most computers employ a removable prepatch problem board system, the most troublesome problem encountered in multishift operation is the setting of the potentiometers and initial conditions. Automatic-setup systems have thus been centered around developing servo-set potentiometers. A servo-set potentiometer system consists of a precision voltage divider and a servo-drive system which can be connected to the shaft of any of the potentiometers. Connection of the servo drive to the potentiometer shaft is made by means of clutches. The potentiometer is set by comparison of the potentiometer voltage with the voltage from the precision voltage divider.

A switching logic similar to the digital voltmeter illustrated in Fig. 1-20 can be used for the precision voltage divider. This voltage divider can be set up either by means of push-button switches or from a punched card or tape input. Selection of the potentiometer to be set can be made by means of a similar device. With this approach, a completely automatic system for computer setup can be developed. As an example, the DO/IT system for the EASE computer can:

1. Set potentiometers automatically from a previously punched and checked paper tape.
2. Read and print the settings of each potentiometer in the computer and simultaneously punch a tape for later reentry.
3. Scan and print out all pertinent voltages within the computer, including:
    (*a*) Amplifier outputs.
    (*b*) Function-generator outputs.
    (*c*) Multiplier outputs.
    (*d*) Miscellaneous equipment.
    (*e*) Interconnecting trunk lines to external equipment.
4. Control the equipment either manually or by means of a punched tape.

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